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NONCOMMUTATIVE JOIN SPACES OF INTEGRAL OPERATORS AND RELATED HYPERSTRUCTURES

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Abstract:

In this contribution we construct noncommutative transposition hypergroups of integral operators on spaces of continuous functions which are determined by Fredholm integral equations of the first and second kinds. We started with integral operators formed by separated kernel. Moreover, we investigate the obtained hyperstructures as transposition hypergroups and also related quasi-hypergroups of blocks of equivalence of integral operators. Moreover, we use also the object function (where the corresponding binary hyperoperation on an ordered group is defined as principal end generated by *products of pairs of elements of the considered group) of a functor enabling the transfer from the category of ordered groups and their isotone homomorphisms into the category of hypergroups and their inclusion homomorphisms.*

The basic group of integral operators contains an invariant subgroup. Using another binary operation on the set of suitable Fredholm integral operators of the second kind we get a group with a significant non-invariant subgroup of operators of the first kind enabling the construction of a quasi-hypergroup of decomposition classes of operators, structure of which is also clarified.

1 Introduction

Contemporary investigations of hyperstructures and their applications yield many relationships and connections between various fields of mathematics.

Besides the motivation for investigation of hyperstructures coming from noncommutative algebra, geometrical structures and other mathematical fields there exist such physical phenomena as the nuclear fission. Nuclear fission occurs when a heavy nucleus, such as U^{235}, splits, or fissions, into two smaller nuclei. As a result of this fission process we can get several dozens of different combinations of two medium-mass elements and several neutrons.

Another typical example of the situation when the result of interaction between two particles is the whole set of particles is the interaction between a foton with certain energy and an electron. The result of this interaction is not deterministic. A photo-electric effect or Coulomb repulsion effect or changeover of foton onto a pair electron and positron can arise.

It is to be noted that a similar situation which occurs during uranium fission appears during several nuclear fission, too. The result depends on conditions. Although the input (2 particles) are the same, the output can be variant.

Another motivation for investigation of hyperstructures yields from technical processes as a time sequence of military car repairs with respect to its roadability consequences and its operational behaviour.

Moreover, some ideas leading naturally to multistructures are also coming from quantum mechanics, quantum optics with applications as quantum cryptography or, in particular, development of quantum computers. The basic idea consist in fact that quantum objects can be simulated in more different states simultaneously.

So, states of quantum object possess the property that they are not spacely localized. Quantum particles are situated in many places at the same time and they are coming through several trajectories simultaneously. Quantum computers give a technology which is of the great interest worldwide. These computers should be able to provide extremely quick computation thanks their possibility to be localized in more states together. In spite of the fact this technology is developed on the basis of single-photon sources it seems to be natural that in the field of quantum communication systems the theory of suitable modified multistructures (hyperstructures—in contemporary terminology) can served as mathematical background. Let us recall the well known physicist *John Archibald Wheeler*, one of coauthors of the "*hypothesis of more worlds*", according to which by the collapse of wave function reality is decomposed on more parallel branches.

1.1 A Scalable Quantum Computer Chip

A scalable quantum computer chip for atomic qubits has been built for the first time by researchers at the University of Michigan, offering hopes for making a practical quantum computer using conventional semiconductor manufacturing technology.

Exploiting the strange rules of the atomic world, quantum computers could potentially break top-secret codes and perform certain kinds of searches much more quickly than conventional computers. The building blocks of quantum computers are called "qubits," or quantum bits, made of such objects as atoms or photons. Connecting multiple qubits via an electrostatic (or other suitable) interaction could then result in a quantum computer, similar to how wiring together individual transistors can make a traditional computer.

Unlike a conventional computer's bits, which can have values of either 0 or 1, a qubit can possess a value of 0 and 1 simultaneously, analogous to a light switch that's on and off at the same time. For their qubit the Michigan group chose an individual cadmium ion, held in free space by a number of electrodes inside a postage-stamp-sized gallium arsenide semiconductor chip. There additional electric fields are able to manipulate the position of the ion, and laser beams could control the qubit value in the ion.

Ions pose an advantage over other potential qubits, such as photons and electron dots, in that they are easier to isolate and shield from external disturbances (noise) that can disrupt their operation. An integrated semiconductor chip is a markedly different environment for ion qubits, which were previously held in hand-made ion traps that could not be easily scaled up or mass produced.

The researchers have not yet demonstrated a quantum computer based on this design, as it only consists of a single qubit. Making a quantum computer would require scaling up a single chip so that it contains enough electrodes to trap many ions simultaneously.

1.2 Fredholm integral equations

In this paper we describe a certain construction (based on a simple but useful lemma—Lemma 1) of hyperstructures belonging to the important class of transposition hypergroups which are also called noncommutative join spaces.

Fredholm integral equations can be considered as a modification of systems of linear equations, thus this topic has algebraic roots. Using operators which correspond to Fredholm equations we will construct ordered groups determining transposition hypergroups. Moreover, using certain subhypergroup of this transposition group we obtain hyperstructures of blocks of operators. In particular, by decomposition of the group of all Fredholm integral operators of the second kind by its subgroup of operators of the first kind we get a quasi-hypergroup of blocks of operators—which is a construction lying in the very foundations of the hyperstructure theory.

An integral equation in the form

$$\varphi(x) - \lambda \int_a^b K(x,s)\varphi(s)\,\mathrm{d}s = f(x), \tag{1}$$

where $K(x,s)$ (kernel), $[x,s] \in \langle a,b\rangle \times \langle a,b\rangle \subset \mathbb{R}\times\mathbb{R}$, is a real or complex valued function (mostly positive real functions), $f(x)$, $x \in \langle a,b\rangle \subset \mathbb{R}$, is a function called a free or an absolute member, λ is a numerical parameter and φ is an unknown function, is called Fredholm integral equation. More precisely, it is called *Fredholm integral equation of the second kind*, whereas integral equation of the form $\int_a^b K(x,s)\varphi(s)\mathrm{d}s = f(x)$ is called *Fredholm integral equation of the first kind*. It is known that under some conditions the solution of the Fredholm integral equation can be expressed both in the form of the sum of Neumann's series and with the usage of Fredholm resolvent.

Usually there are considered Fredholm integral equations with a nondegenerate Lebesgue square integrable kernel $K(x,s)$, i.e. Lebesgue integral $\iint_M |K(x,s)|^2\,\mathrm{d}x\mathrm{d}s$, where $M = \langle a,b\rangle \times \langle a,b\rangle \subset \mathbb{R}\times\mathbb{R}$, is convergent. In this contribution we will construct noncommutative join spaces on the set of operators $F(\lambda, K, f)$ with continuous functions f, K (absolute member and kernel) and a nonzero parameter λ. For our purposes we will consider continuous positive functions only, in order to avoid some obstacles in integrability of functions in the form of fractions. So we consider operators $F(\lambda, K, f)\colon C\langle a,b\rangle \to C\langle a,b\rangle$ ($C\langle a,b\rangle$ means the set of continuous functions on $\langle a,b\rangle$) of the type

$$F(\lambda, K, f)(\varphi(x)) = \lambda \int_a^b K(x,s)\varphi(s)\,\mathrm{d}s + f(x) \tag{2}$$

with a fixed interval $\langle a,b\rangle \subset \mathbb{R}$. The mentioned operator occurs in the construction of a series of functions which approximate the solution of Fredholm equation (1).

2 Preliminaries

Now, recall some basic notions and used denotation from the hypergroup theory—[1, 2, 4]. We remind that a *hypergroupoid* is a pair $(H, \bullet)$, where $H \neq \emptyset$ and operation $\bullet : H \times H \to \mathbb{P}^*(H)$ (the system of all nonempty subsets of H) is a binary hyperoperation on H. If the associativity axiom $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ holds for all $a, b, c \in H$ than the pair $(H, \bullet)$ is called a *semihypergroup*. If moreover the reproduction axiom $a \bullet H = H = H \bullet a$ is satisfied for any element $a \in H$ than the pair $(H, \bullet)$ is called a *hypergroup*. Here for any pair of nonempty subsets $A, B \subseteq H$ we define its *hyperproduct* as $A \bullet B = \bigcup\{a \bullet b\,; a \in A, b \in B\}$. A hypergroupoid $(H, \bullet)$ where the reproduction axiom is fulfilled is called a *quasi-hypergroup*. A *subhypergroupoid* of a hypergroupoid $(H, \bullet)$ is a pair $(S, \bullet)$, where $S \bullet S \subseteq S$, i.e. the set S is multiplicatively closed. If the subhypergroupoid $(S, \bullet)$ of $(H, \bullet)$ is a hypergroup then it is said to be a *subhypergroup* of $(H, \bullet)$.

A hypergroup $(H, \bullet)$ is called a *transposition hypergroup* or a *join space* if it satisfies the transposition axiom: For all $a, b, c, d \in H$ the relation $b\backslash a \approx c/d$ implies $a \bullet d \approx b \bullet c$, (here $X \approx Y$ for $X, Y \subseteq H$ means $X \cap Y \neq \emptyset$), where sets $b\backslash a = \{x \in H; a \in b \bullet x\}$, $c/d = \{x \in H; c \in x \bullet d\}$ are called *left and right extensions or fraction*, respectively.

We describe first the simple but important construction from [4] which has been used also in [2] and enables to obtain in a certain sense analogous results to those presented in this contribution.

By an *quasi-ordered semigroup* we mean a triple $(G, \bullet, \leqq)$, where $(G, \bullet)$ is a semigroup and binary relation "$\leqq$" is a quasiorder (reflexive and transitive) on the set G such that for any triple $x, y, z \in G$ with the property $x \leqq y$ also $x \bullet z \leqq y \bullet z, z \bullet x \leqq z \bullet y$.

By an *ordered (semi)group* we mean (as usual) a triple $(G, \bullet, \leqq)$, where $(G, \bullet)$ is a (semi)group and "$\leqq$" is a reflexive, antisymmetrical and transitive binary relation on the set G such that for any triple $x, y, z \in G$ with the property $x \leqq y$ also $x \bullet z \leqq y \bullet z, z \bullet x \leqq z \bullet y$. Further, $[a)_{\leqq} = \{x \in G; a \leqq x\}$ is a *principal end* generated by $a \in G$.

The following lemma which is crucial for our considerations is proved in [7]; firstly in [4, p. 146, 147].

Lemma 1. *Let a triple $(G, \cdot, \leqq)$ be a quasi-ordered semigroup. Define a hyperoperation $*\colon G \times G \to \mathbb{P}^*(G)$ by $a * b = [a \cdot b)_{\leqq} = \{x \in G; a \cdot b \leqq x\}$ for all pairs of elements $a, b \in G$.*

1. *Then $(G, *)$ is a semihypergroup which is commutative if the semigroup $(G, \cdot)$ is commutative.*
2. *Let $(G, *)$ be the above defined semihypergroup. Then $(G, *)$ is a hypergroup iff for any pair of elements $a, b \in G$ there exists a pair of elements $c, c' \in G$ with a property $a \cdot c \leqq b$, $c' \cdot a \leqq b$.*

Remark 1. In case the binary relation "$\leqq$" is even ordering the commutativity of $(G, *)$ implies the commutativity of $(G, \cdot)$.(see [4, p. 146, 147])

Corollary 1. Let $(G, \cdot, \leqq)$ be an ordered group. Define a hyperoperation "$*$" as follows: $*\colon G \times G \to \mathbb{P}^*(G)$ by $a * b = [a \cdot b)_{\leqq} = \{x \in G; a \cdot b \leqq x\}$ for all pairs of elements $a, b \in G$. Then $(G, *)$ is a hypergroup which is commutative if and only if the group $(G, \cdot)$ is commutative.

3 Separable kernels

As quoted above, linear integral equations are the continuous analog of systems of algebraic equations. From this point of view it is to be noted that Fredholm theory is the theory of integral equations that have kernels $K(x, s)$ that can be approximated arbitrarily accurately by separable kernels. These kernels can be written in the form $K(x, s) = \sum_{i=1}^{n} f_i(x) \cdot g_i(s)$. It is useful to suppose that the sum has been reduced to a form in which the functions f_i are linearly independent and the functions g_i are also linearly independent. Using [11] let us introduce simple example. Let us consider the equation

$$u(x) = f(x) + c \int_{-1}^{1} \cos \pi(x - y) u(y) \mathrm{d}y, \tag{3}$$

where f(x) is some given integrable function. By expanding the cosine we can rewrite this as

$$u(x) = f(x) + cu_1 \sin(\pi x) + cu_2 \cos(\pi x), \tag{4}$$

where $u_1 = \int_{-1}^{1} \sin(\pi y) u(y) \mathrm{d}y$, $u_2 = \int_{-1}^{1} \cos(\pi y) u(y) \mathrm{d}y$. Multiplying the equation by $\sin(\pi x)$ and integrating over the interval, and then doing the same with $\cos(\pi x)$, gives

$$u_1 = f_1 + cu_1, \qquad u_2 = f_2 + cu_2. \tag{5}$$

When $c = 1$, there are two possibilities. If either f_1 or f_2 is not zero, the algebraic system (5) has no solution, so the integral equation (3) has no solution. The alternative is that $f_1 = f_2 = 0$. These are the *Fredholm conditions* on $f(x)$. If they are satisfied, then (with $c = 1$) the algebraic equation (5) is satisfied whatever values u_1 and u_2 may be. Thus when $c = 1$, either (3) has no solution at all or, if the Fredholm conditions are satisfied, there is a solution but it is not unique.

The kernel in this problem is *separable*. For any separable kernel, the problem boils down to a matter of linear algebraic equations.

In the second part in connection with the Fredholm integral equations of first kind we will study a special subgroup G_0 so called second group of Fredholm Integral Operators.

The mentioned group G_0 consists of operators of the form $F(1, K, 0)$ where K is a positive functions continuous on the interval $J \times J$ and this operator is acting in such a way that for any continuous function $\varphi \in C(J)$ we have

$$F(1, k, 0)(\varphi) = \int_a^b K(x, s)\varphi(s) \mathrm{d}s.$$

As a motivating construction for chapter 5.2 we describe a construction of action of a hypergroupoid formed by integral operators with a separable kernels on the state set of positive continuous functions.

In other words, it is so called multiautomaton with input alphabet formed by a hypergroupoid of just mentioned operators. This concept consists of a certain generalization of a so called quasi-automaton where the condition MAC is replaced by GMAC (Generalized Mixed Associativity Condition)—see [1].

For an arbitrary fixed positive integer $n \in N$ we denote by $Sp_n(J \times J)$ the set of all functions of the form $K(x,s) = \sum_{i=1}^{n} k_{1i}(x)k_{2i}(s)$, $k_{1i}, k_{2i} \in \mathbb{C}(J)$, $i = 1, 2\dots, n$ which are in the role of separable kernels of the same length n.

We will construct a hypergroupoid $(Sp_n(J \times J), \circ)$ acting on the ring of continuous functions $\mathbb{C}(J)$. For any pair $K(x,s) = \sum_{i=1}^{n} k_{1i}(x)k_{2i}(s)$, $G(x,s) = \sum_{i=1}^{n} g_{1i}(x)g_{2i}(s)$ of functions from $Sp_n(J \times J)$ we define the hyperproduct $K \circ G$ as the set of all functions $H(x,s) = \sum_{j=1}^{n} h_{1j}(x)h_{2j}(s)$, $h_{1j}, h_{2j} \in \mathbb{C}(J)$, such that $F(1,H,0)(\varphi) = \int_a^b H(x,s)\varphi(s)\,\mathrm{d}s = \sum_{j=1}^{n} c_j h_{1j}(x)$, $c_j = \int_a^b h_{2j}(s)\varphi(s)\,\mathrm{d}s$, where $g_{1j}(x) \leqq h_{1j}(x)$ for all $j = 1, 2\dots, n$, $x \in J$ and $\sum_{i=1}^{n} a_i b_{ij} \leqq c_j$, $a_i = \int_a^b k_{2i}(x)\varphi(x)\,\mathrm{d}x$, $b_{ij} = \int_a^b k_{1i}(x)g_{2j}(x)\,\mathrm{d}x$, $i, j = 1, 2\dots, n$, thus a_i, b_{ij} are scalar products $a_i = (k_{2i}, \varphi)$, $b_{1i} = (k_{1i}, g_{2j})$. Evidently, $\big(Sp_n(J \times J), \circ\big)$ is a non-commutative hypergroupoid of separable kernels of integral operators $F(1,H,0)$, $K \in Sp_n(J \times J)$.

Now, we define the transition function $\delta_j : \mathbb{C}(J) \times Sp_n(J \times J) \to \mathbb{C}(J)$ in this way: For any $K(x,s) = \sum_{i=1}^{n} k_{1i}(x)k_{2i}(s) \in Sp_n(J \times J)$ and any $\varphi \in \mathbb{C}(J)$ we put

$$\begin{aligned}\delta_j(\varphi, k) = F(1,H,0)(\varphi) &= \int_a^b K(x,s)\varphi(s)\,\mathrm{d}s \\ &= \sum_{i=1}^{n} k_{1i}(x)\int_a^b k_{2i}(s)\varphi(s)\,\mathrm{d}s = \sum_{i=1}^{n} a_i k_{1i}(x). \qquad (6)\end{aligned}$$

with coefficient $a_i = \int_a^b k_{2i}(s)\varphi(s)\,\mathrm{d}s = (k_{2i}, \varphi)$, $i = 1, 2\dots, n$. We conclude this part of our considerations by verification that the hypergroupoid $\big(Sp_n(J \times J), \circ\big)$ really acts on the ring $\mathbb{C}(J)$, i.e. that the Generalized Mixed Associativity Condition (GMAC) is satisfied. More in details we show that any triad

$$[K, G, \varphi] \in Sp_n(J \times J) \times Sp_n(J \times J) \times \mathbb{C}(J)$$

the relationship $\delta_j\big(\delta_j(\varphi, k), G\big) \in \delta_j(\varphi, k \circ G) = \{\delta_j(\varphi, H);\, H \in K \circ G\}$ holds.

Indeed, for arbitrary $K(x,s) = \sum_{i=1}^{n} k_{1i}(x)k_{2i}(s)$, $G(x,s) = \sum_{i=1}^{n} g_{1i}(x)g_{2i}(s)$ and any function $\varphi \in \mathbb{C}(J)$ we have

$$\begin{aligned}\delta_j\big(\delta_j(\varphi, K), G\big) &= \delta_j\Big(\sum_{i=1}^{n} a_i k_{2i}(x), G\Big) = \int_a^b \Big(\sum_{i=1}^{n} g_{1i}(x)\int_a^b g_{2i}(s)\Big)\Big(\sum_{i=1}^{n} a_i k_{2i}(x)\Big)\,\mathrm{d}s \\ &= \int_a^b \big(g_{11}(x)g_{21}(s) + g_{12}(x)g_{22}(s) + \dots + g_{1n}(x)g_{2n}(s)\big)\cdot\end{aligned}$$

$$
\begin{aligned}
&\cdot \big(a_1 k_{11}(s) + a_2 k_{12}(s) + \dots + a_n k_{1n}(s)\big)\,\mathrm{d}s\\
&= \int_a^b \big(a_1 g_{11}(x) g_{21}(s) k_{11}(s) + a_1 g_{12}(x) g_{22}(s) k_{11}(s) + \dots + a_1 g_{1n}(x) g_{2n}(s) k_{11}(s)\\
&\quad + a_2 g_{11}(x) g_{21}(s) k_{12}(s) + \dots + a_2 g_{1n}(x) g_{2n}(s) k_{12}(s) + \dots\\
&\quad + a_n g_{11}(x) g_{21}(s) k_{1n}(s) + \dots + a_n g_{1n}(x) g_{2n}(s) k_{1n}(s)\big)\,\mathrm{d}s\\
&= a_1 g_{11}(x) \int_a^b g_{21}(s) k_{11}(s)\,\mathrm{d}s + a_1 g_{12}(x) \int_a^b g_{22}(s) k_{11}(s)\,\mathrm{d}s + \dots\\
&\quad + a_1 g_{1n}(x) \int_a^b g_{2n}(s) k_{11}(s)\,\mathrm{d}s + a_2 g_{11}(x) \int_a^b g_{21}(s) k_{12}(s)\,\mathrm{d}s + \dots\\
&\quad + a_2 g_{12}(x) \int_a^b g_{22}(s) k_{12}(s)\,\mathrm{d}s + \dots + a_2 g_{1n}(x) \int_a^b g_{2n}(s) k_{12}(s)\,\mathrm{d}s + \dots\\
&\quad + a_n g_{11}(x) \int_a^b g_{21}(s) k_{1n}(s)\,\mathrm{d}s + \dots + a_n g_{1n}(x) \int_a^b g_{2n}(s) k_{1n}(s)\,\mathrm{d}s\\
&= a_1 b_{11} g_{11}(x) + a_1 b_{12} g_{12}(x) + \dots + a_1 b_{1n} g_{1n}(x) + a_2 b_{21} g_{11}(x) + a_2 b_{22} g_{12}(x)\\
&\quad + \dots + a_2 b_{2n} g_{1n}(x) + \dots + a_n b_{n1} g_{11}(x) + \dots + a_n b_{nn} g_{1n}(x)\\
&= (a_1 b_{11} + a_2 b_{21} + \dots + a_n b_{n1}) g_{11}(x) + (a_1 b_{12} + a_2 b_{22} + \dots + a_n b_{n2}) g_{12}(x)\\
&\quad + \dots + (a_1 b_{1n} + a_2 b_{2n} + \dots + a_n b_{nn}) g_{1n}(x)\\
&= \Big(\sum_{i=1}^n a_i b_{i1}\Big) g_{11}(x) + \Big(\sum_{i=1}^n a_i b_{i2}\Big) g_{12}(x) + \dots + \Big(\sum_{i=1}^n a_i b_{in}\Big) g_{1n}(x)\\
&= \sum_{j=1}^n \Big(\sum_{i=1}^n a_i b_{ij}\Big) g_{1j}(x) \in \Big\{\sum_{j=1}^n c_j h_{1j}(x);\ \sum_{i=1}^n a_i b_{ij} \leqq c_j,\ g_{1j}(x) \leqq h_{1j}(x),\\
&h_{1j} \in \mathbb{C}(J),\ j = 1, 2, \dots, n,\ x \in J\Big\} = \{H(x,s)(\varphi);\ H \in K \circ G\} = \delta_j(\varphi, K \circ G),
\end{aligned}
$$

therefore GMAC (the Generalized Mixed Associativity Condition) is satisfied. Consequently, $(\mathbb{C}(J), S_{p_n}(JxJ), \delta_j)$ is a multiautomaton with the state set $\mathbb{C}(J)$ and the input hypergroupoid $(S_{p_n}(JxJ), \circ)$ of separable kernels

$$K(x,s) = \sum_{i=1}^n k_{1i}(x) k_{2i}(y) \in \mathbb{C}(J \times J)$$

with the above defined binary hyperoperation, i.e. it is an action of the mentioned hypergroupoid on the set $\mathbb{C}(J)$.

It is to be noted that multiautomata constructed from integral operators are of the so called "centralizer" type []. It means that input hyperstructures are constructed using centralizers of given operators. Here, using separable kernels we obtained construction of multiautomata of more general form.

4 Constructions of join spaces of operators based on ordered groups

In the sequel we will denote $C(J)$, $C(J \times J)$ the sets of continuous functions on J, $J \times J$, respectively, where $J \subseteq \mathbb{R}$ is an interval and $f(x) \neq 0$ for all $x \in J$.

Proposition 1. Let $J = \langle a, b\rangle$, $\mathbb{F} = \{F(\lambda, K, f) : K(x,s) \in C(J \times J),\ f \in C_+(J),\ \lambda \neq 0\}$, where $F(\lambda, K, f)$ is given by (2). For any pairs of operators $F(\lambda_1, K_1, f_1)$, $F(\lambda_2, K_2, f_2)$ in $\mathbb{F}$ let us define

$$F(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2) = F(\lambda_1\lambda_2, K_2 f_1 + K_1, f_1 f_2) \tag{7}$$

and $F(\lambda_1, K_1, f_1) \leqq F(\lambda_2, K_2, f_2)$ if and only if $\lambda_1 = \lambda_2$, $f_1(x) \equiv f_2(x)$ and $K_1(x,s) \leqq K_2(x,s)$ for any $[x,s] \in J \times J$. Then $(\mathbb{F}, \cdot, \leqq)$ is a noncommutative ordered group.

Proof. The proof is straightforward. □

Now we apply the simple construction of a hypergroup from Lemma 1 onto this considered concrete case of integral operators:

For an arbitrary pair of operators $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2) \in \mathbb{F}$ we define a hyperoperation $*\colon \mathbb{F} \times \mathbb{F} \to \mathbb{P}^*(\mathbb{F})$ as follows:

$$\begin{aligned} F(\lambda_1, &K_1, f_1) * F(\lambda_2, K_2, f_2) = \\ &= \{F(\lambda, K, f) \in \mathbb{F};\ F(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2) \leqq F(\lambda, K, f)\} \\ &= \{F(\lambda, K, f) \in \mathbb{F};\ F(\lambda_1\lambda_2, K_2 f_1 + K_1, f_1 f_2) \leqq F(\lambda, K, f)\} \\ &= \{F(\lambda_1\lambda_2, K, f_1 f_2);\ K_2(x,s) f_1(x) + K_1(x,s) \leqq K(x,s), [x,s] \in J \times J\}. \end{aligned} \tag{8}$$

Then we obtain from Proposition 1 with respect to Lemma 1 immediately:

Proposition 2. Let $J = \langle a, b\rangle \subseteq \mathbb{R}$ and $*\colon \mathbb{F} \times \mathbb{F} \to \mathbb{P}^*(\mathbb{F})$ be the above defined binary hyperoperation. Then the hypergroupoid $(\mathbb{F}, *)$ is a noncommutative hypergroup.

Now we are going to verify that the above constructed noncommutative hypergroup $(\mathbb{F}, *)$ is in fact a join space. The following auxiliary assertion will be very useful for the proof.

Lemma 2. *Let $J \subseteq \mathbb{R}$ be a compact interval and $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2) \in \mathbb{F}$ be arbitrary operators, i.e. elements of the hypergroup $(\mathbb{F}, *)$. Then*

1° $F(\lambda_1, K_1, f_1)/F(\lambda_2, K_2, f_2) =$
$$= \left\{ F\Big(\frac{\lambda_1}{\lambda_2}, K, \frac{f_1}{f_2}\Big) : K(x,s) \leqq K_1(x,s) - K_2(x,s)\,\frac{f_1(x)}{f_2(x)},\ [x,s] \in J \right\},$$

2° $F(\lambda_2, K_2, f_2)\backslash F(\lambda_1, K_1, f_1) =$
$$= \left\{ F\Big(\frac{\lambda_1}{\lambda_2}, K, \frac{f_1}{f_2}\Big) : K(x,s) \leqq \frac{K_1(x,s) - K_2(x,s)}{f_2(x)},\ [x,s] \in J \right\}.$$

Proof. Considering the fact that the function f_2 is positive and $\lambda_2 \neq 0$ on the whole $J \times J$ with respect to the definitions of the corresponding hyperoperations we obtain for arbitrary pairs of operators $F(\lambda_1, K_1, f_1)$, $F(\lambda_2, K_2, f_2) \in \mathbb{F}$ that

$$F(\lambda_1, K_1, f_1)/F(\lambda_2, K_2, f_2) =$$
$$= \left\{ F\Big(\frac{\lambda_1}{\lambda_2}, K, \frac{f_1}{f_2}\Big) : K(x,s) \leqq K_1(x,s) - K_2(x,s)\frac{f_1(x)}{f_2(x)} \right\}.$$

Hence formula 1° is proved. Finally,

$$F(\lambda_2, K_2, f_2)\backslash F(\lambda_1, K_1, f_1) =$$
$$= \left\{ F\Big(\frac{\lambda_1}{\lambda_2}, K, \frac{f_1}{f_2}\Big) : K(x,s) \leqq \frac{K_1(x,s) - K_2(x,s)}{f_2(x)} \right\}.$$

and formula 2° is proved, as well. □

Theorem 1. Let $J \times J \subseteq \mathbb{R} \times \mathbb{R}$, $\mathbb{F} = \{F(\lambda, K, f) : K \in C(J \times J),\ f \in C_+(J),\ \lambda \neq 0\}$ be the set of Fredholm integral operators. If $F(\lambda_1, K_1, f_1) * F(\lambda_2, K_2, f_2) = \{F(\lambda, K, f) \in \mathbb{F}; \lambda_1\lambda_2 = \lambda,\ f_1 f_2 = f,\ K_2 f_1 + K_1 \leqq K\}$ for any pair $F(\lambda_1, K_1, f_1)$, $F(\lambda_2, K_2, f_2) \in \mathbb{F}$, then $(\mathbb{F}, *)$ is a noncommutative transposition hypergroup, i.e. a noncommutative join space.

Proof. By Proposition 2 the hypergroupoid $(\mathbb{F}, *)$ is a noncommutative hypergroup. It remains to prove that this hypergroup satisfies the transposition law:

Suppose $F(\lambda_i, K_i, f_i) \in \mathbb{F}$, $i = 1, 2, 3, 4$ is a quadruple of integral operators such that
$$F(\lambda_2, K_2, f_2)\backslash F(\lambda_1, K_1, f_1) \approx F(\lambda_3, K_3, f_3)/F(\lambda_4, K_4, f_4),$$
then $F(\lambda_1, K_1, f_1) * F(\lambda_4, K_4, f_4) \approx F(\lambda_2, K_2, f_2) * F(\lambda_3, K_3, f_3)$. If

$$\left\{ F\Big(\frac{\lambda_1}{\lambda_2}, K, \frac{f_1}{f_2}\Big) : K \leqq \frac{K_1 - K_2}{f_2} \right\} \cap \left\{ F\Big(\frac{\lambda_3}{\lambda_4}, K, \frac{f_3}{f_4}\Big) : K \leqq K_3 - K_4\,\frac{f_3}{f_4} \right\} \neq \emptyset,$$

thus there exist an operator $F(\lambda, K, f) \in \mathbb{F}$ such that $\lambda = \dfrac{\lambda_1}{\lambda_2} = \dfrac{\lambda_3}{\lambda_4}$ and $f = \dfrac{f_1}{f_2} = \dfrac{f_3}{f_4}$ we have $\lambda_1\lambda_4 = \lambda_2\lambda_3$, $f_1 f_4 = f_2 f_3$ and K is a function satisfying

$K \leqq \frac{K_1-K_2}{f_2}, K \leqq K_3 - K_4 \frac{f_3}{f_4}$. Let us define $\lambda = \lambda_1\lambda_4 = \lambda_2\lambda_3$, $f(x) = f_1(x)f_4(x) = f_2(x)f_3(x)$, $x \in J$ and

$$K(x,s) \geqq \max\{K_4(x,s)f_1(x) + K_1(x,s), K_3(x,s)f_2(x) + K_2(x,s)\}, x, s \in J.$$

Then $F(\lambda, K, f) \in \mathbb{F}$ and with respect to Lemma 2 we have

$$F(\lambda, K, f) \in \{F(\lambda_1\lambda_4, K, f_1f_4); K_4f_1 + K_1 \leqq K\} = F(\lambda_1, K_1, f_1) * F(\lambda_4, K_4, f_4),$$

$$F(\lambda, K, f) \in \{F(\lambda_2\lambda_3, K, f_2f_3); K_3f_2 + K_2 \leqq K\} = F(\lambda_2, K_2, f_2) * F(\lambda_3, K_3, f_3),$$

consequently $F(\lambda_1, K_1, f_1) * F(\lambda_4, K_4, f_4) \approx F(\lambda_2, K_2, f_2) * F(\lambda_3, K_3, f_3)$, hence the hypergroup $(\mathbb{F}, *)$ is a noncommutative join space. □

5 Constructions of join spaces of operators based on decompositions

The following simple general construction will be used in special cases for groups of integral operators.

Lemma 3. *Let $\boldsymbol{R}$ be an equivalence on an arbitrary set S. For $x, s \in S$ let us define a hyperoperation $\star\colon S \times S \to \mathbb{P}^*(S)$ as follows:*

$$x \star y = \mathscr{X} \cup \mathscr{Y}, \quad \text{where } \mathscr{X}, \mathscr{Y} \text{ are the classes of decomposition } S/\boldsymbol{R} \text{ containing } x, y, \text{ respectively.}$$

Then the pair $(S, \star)$ is a commutative join space.

Proof. For an arbitrary equivalence $\boldsymbol{R}$ on the set S and for the corresponding decomposition $S/\boldsymbol{R}$ by defining $x \star y = \mathscr{X} \cup \mathscr{Y}$ for $x, y \in S$, $x \in \mathscr{X}$, $y \in \mathscr{Y}$ ($\mathscr{X}, \mathscr{Y}$ not necessarily different) we get that $(S, \star)$ is a commutative hypergroup. Indeed, if $x \in \mathscr{X}$, $y \in \mathscr{Y}$, $z \in \mathscr{Z}$, where $\mathscr{X}, \mathscr{Y}, \mathscr{Z} \in S/\boldsymbol{R}$, then

$$x \star (y \star z) = x \star (\mathscr{Y} \cup \mathscr{Z}) = (x \star \mathscr{Y}) \cup (x \star \mathscr{Z}) = \Big(\bigcup_{t \in \mathscr{Y}} x \star t\Big) \cup \Big(\bigcup_{u \in \mathscr{Z}} x \star u\Big) =$$
$$(\mathscr{X} \cup \mathscr{Y}) \cup (\mathscr{X} \cup \mathscr{Z}) = \mathscr{X} \cup \mathscr{Y} \cup \mathscr{Z} = \mathscr{Z} \cup \mathscr{X} \cup \mathscr{Y} = z \star (x \star y) = (x \star y) \star z.$$

It remains to prove that the reproduction axiom holds. Indeed,

$$x \star S = S \star x = \bigcup_{t \in S} t \star x = \bigcup_{t \in S} \mathscr{X} \star \boldsymbol{R}(t) = \bigcup S/\boldsymbol{R} = S.$$

For $\alpha, \beta \in S$ we have

$$\alpha/\beta = \{x \in S, \alpha \in x \star \beta\} = \begin{cases} \boldsymbol{R}(\alpha) & \text{if } \alpha \text{ non} \boldsymbol{R}\, \beta, \\ S & \text{if } \alpha\, \boldsymbol{R}\, \beta. \end{cases}$$

Here $\boldsymbol{R}(\alpha) = \{\gamma \in S, \alpha\,\boldsymbol{R}\,\gamma\}$. It remains to prove the transposition axiom, i.e. $\alpha/\beta \approx \gamma/\delta$ implies $\alpha \star \delta \approx \beta \star \gamma$, therefore $\boldsymbol{R}(\alpha) \cup \boldsymbol{R}(\delta) \approx \boldsymbol{R}(\beta) \cup \boldsymbol{R}(\gamma)$ in our case.

1. Let $\alpha/\beta = \boldsymbol{R}(\alpha)$ and $\gamma/\delta = \boldsymbol{R}(\gamma)$. Then $\boldsymbol{R}(\alpha) \approx \boldsymbol{R}(\gamma)$ implies $\alpha\,\boldsymbol{R}\,\gamma$, thus $\boldsymbol{R}(\alpha) = \boldsymbol{R}(\gamma)$, thus $\alpha \star \delta \approx \beta \star \gamma$ holds.
2. Let $\alpha/\beta = \boldsymbol{R}(\alpha)$ and $\gamma/\delta = S$. Then $\boldsymbol{R}(\gamma) = \boldsymbol{R}(\delta)$ and $\alpha \star \delta \approx \beta \star \gamma$ holds.

The remaining cases can be verified in a similar way. □

5.1 First group of Fredholm integral operators

Let us denote by $\mathbb{F}_1, \mathbb{F}_2$ subsets of $\mathbb{F}$ formed by integral operators of the form $F(1, K, f)$, $F(\lambda, K, 1)$, respectively.

Lemma 4. *The above defined sets $\mathbb{F}_1, \mathbb{F}_2$ are the carriers of normal subgroups of the group $\mathbb{F}$.*

Proof. Let $F(1, K, f)$, $F(1, K_1, f_1) \in \mathbb{F}_1$ be an arbitrary pair of operators. Then

$$F(1, K, f) \cdot F^{-1}(1, K_1, f_1) = F\Big(1, K - K_1\frac{f}{f_1}, \frac{f}{f_1}\Big) \in \mathbb{F}_1, \tag{9}$$

hence $(\mathbb{F}_1, \cdot)$ is a subgroup of $(\mathbb{F}, \cdot)$. Similarly for $F(\lambda, K, 1)$, $F(\bar{\lambda}, \bar{K}, 1) \in \mathbb{F}_2$:

$$F(\lambda, K, 1) \cdot F^{-1}(\bar{\lambda}, \bar{K}, 1) = F(\lambda, K, 1) \cdot F\Big(\frac{1}{\bar{\lambda}}, -\bar{K}, 1\Big) = F\Big(\frac{\lambda}{\bar{\lambda}}, K - \bar{K}, 1\Big) \in \mathbb{F}_2.$$

So, $(\mathbb{F}_2, \cdot)$ is a subgroup of the group $(\mathbb{F}, \cdot)$. Further, let $F(\lambda, K, f) \in \mathbb{F}$, $F(1, K_1, f_1) \in \mathbb{F}_1$. Then

$$\begin{aligned} &F(\lambda, K, f) \cdot F(1, K_1, f_1) \cdot F^{-1}(\lambda, K, f) = \\ &\quad = F(\lambda, K_1 f + K, f f_1) \cdot F\Big(\frac{1}{\lambda}, -\frac{K}{f}, \frac{1}{f}\Big) = F\big(1, K_1 f - K(f_1 - 1), f_1\big) \in \mathbb{F}_1. \end{aligned}$$

Thus, for an arbitrary $F(\lambda, K, f) \in \mathbb{F}$ we obtain $F(\lambda, K, f) \cdot \mathbb{F}_1 \cdot F^{-1}(\lambda, K, f) \subset \mathbb{F}_1$. Consequently the group $(\mathbb{F}_1, \cdot)$ is a normal subgroup of the group $(\mathbb{F}, \cdot)$.

Analogously let $F(\lambda, K, f) \in \mathbb{F}$, $F(\bar{\lambda}, \bar{K}, 1) \in \mathbb{F}_2$ be an arbitrary pair of operators. Then $F(\lambda, K, f) \cdot F(\bar{\lambda}, \bar{K}, 1) \cdot F^{-1}(\lambda, K, f) = F(\bar{\lambda}, \bar{K} f, 1) \in \mathbb{F}_2$. We get that the group $(\mathbb{F}_2, \cdot)$ is a normal subgroup of the group $(\mathbb{F}, \cdot)$. □

5.2 Second group of Fredholm integral operators

Let us define another type of multiplication of Fredholm integral operators. We consider the set $\mathbb{G} = \{F(\lambda, K, f), \lambda \neq 0, K \neq 0$ for arbitrary pairs $(x, y) \in J \times J$, $K \in C(J \times J)$, $f \in C(J)\}$.

$$F(\lambda_1, K_1, f_1) \odot F(\lambda_2, K_2, f_2) = F(\lambda_1\lambda_2, K_1K_2, \lambda_1\hat{K}_1 f_2 + f_1), \quad \text{where } \hat{K}(x,s) = K(x,x) \tag{10}$$

(we use only a diagonal of the square $\langle a, b\rangle \times \langle a, b\rangle$). The $(\mathbb{G}, \odot)$ is a noncommutative group of Fredholm operators of the second kind which has a subgroup of the form $\mathbb{G}_0 = \{F(1, K, 0); K \in C_+(J \times J)\}$.

Here $C_+(J \times J)$ means the set of positive continuous functions on $J \times J$.

5.3 Decomposition of join spaces determined by subgroups $\mathbb{F}_1$, $\mathbb{F}_2$ and $\mathbb{G}_0$

This part is devoted to the algebraization of the left decomposition of the group $(\mathbb{G}, \odot)$ by its (non–invariant) subgroup $(\mathbb{G}_0, \odot)$ under certain simplified conditions concerning kernels and free members determining considered integral operators of the form

$$F(1, K, 0)(\varphi) = \int_a^b K(x, s)\varphi(s)\mathrm{d}s,$$

which can be specified under some changes concerning bounds of the considered integral and by transfer into the complex domain. Concretely, by special choice of kernels $K(x, y)$ we obtain various integral transformations as classical Laplace or Laplace-Carson transforms, Fourier transforms or Mellin and Hankel transforms. Of course, mentioned integral transformations are well-defined under special conditions demanded for pre-image functions φ.

We describe left and right decompositions of $(\mathbb{G}, \odot)$ determined by its subgroup $(\mathbb{G}_0, \odot)$ which are created by the left and right translation of this subgroup,

$$\mathscr{L}(\mathbb{G}) = \mathbb{G}/_L\mathbb{G}_0 = \{F(\lambda, K, f) \odot \mathbb{G}_0; F(\lambda, K, f) \in \mathbb{G}\},$$
$$\mathscr{R}(\mathbb{G}) = \mathbb{G}/_R\mathbb{G}_0 = \{\mathbb{G}_0 \odot F(\lambda, K, f); F(\lambda, K, f) \in \mathbb{G}\}.$$

Here

$$\boldsymbol{E}_L = \left\{[F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2)] \in \mathbb{G} \times \mathbb{G};\ F^{-1}(\lambda_1, K_1, f_1) \odot F(\lambda_2, K_2, f_2) \in \mathbb{G}_0\right\},$$
$$\boldsymbol{E}_R = \left\{[F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2)] \in \mathbb{G} \times \mathbb{G};\ F(\lambda_2, K_2, f_2) \odot F^{-1}(\lambda_1, K_1, f_1) \in \mathbb{G}_0\right\}$$

are the corresponding equivalences. Evidently, we have $F(\lambda_1, K_1, f_1)\, \boldsymbol{E}_L\, F(\lambda_2, K_2, f_2)$ if and only if $F(\lambda_2, K_2, f_2) \in F(\lambda_1, K_1, f_1) \odot \mathbb{G}_0$, i.e. $\exists F(1, K, 0) \in \mathbb{G}_0$ such that for a suitable kernel $K(x, s) : \langle a, b\rangle \times \langle a, b\rangle \to \mathbb{R}$ determining the operator $\varphi(x) \mapsto \int_a^b K(x, s)\varphi(s)\,\mathrm{d}s$ it holds: $F(\lambda_1, K_1, f_1) \odot F(1, K, 0) = F(\lambda_2, K_2, f_2)$, so $F(\lambda_1, KK_1, f_1) = F(\lambda_2, K_2, f_2)$ and $\lambda_1 = \lambda_2$, $KK_1 = K_2$, $f_1 = f_2$.

Whereas $F(\lambda_1, K_1, f_1)\, \boldsymbol{E}_R\, F(\lambda_2, K_2, f_2)$ if and only if $F(\lambda_2, K_2, f_2) \in \mathbb{G}_0 \odot F(\lambda_1, K_1, f_1)$, $F(\lambda_2, K_2, f_2) = F(\lambda_1, KK_1, \hat{K} f_1)$ for suitable $F(1, K, 0) \in \mathbb{G}_0$, i.e. $\exists F(1, K, 0) \in \mathbb{G}_0$ such that $K_2 = KK_1, \lambda_1 = \lambda_2, f_2 = \hat{K} f_1$.

Let us describe the decomposition of the group $(\mathbb{F}, \cdot)$ determined by its subgroup $(\mathbb{F}_1, \cdot)$. As it was mentioned above the subgroup is normal, so the left and right decompositions are equal.

$$\mathbb{F}/_R\mathbb{F}_1 = \mathbb{F}/_L\mathbb{F}_1 = \{\mathbb{F}_1 \cdot F(\lambda, K, f) = F(\lambda, K, f) \cdot \mathbb{F}_1;\, F(\lambda, K, f) \in \mathbb{F}\}.$$

Here

$$\boldsymbol{E_1} = \{[F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2)] \in \mathbb{F} \times \mathbb{F};\; F^{-1}(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2) \in \mathbb{F}_1 \\ \text{or } F(\lambda_2, K_2, f_2) \cdot F^{-1}(\lambda_1, K_1, f_1) \in \mathbb{F}_1\}$$

is the corresponding equivalence.

Evidently, we have $F(\lambda_1, K_1, f_1)\, \boldsymbol{E_1}\, F(\lambda_2, K_2, f_2)$ if and only if $F(\lambda_2, K_2, f_2) \in \mathbb{F}_1 \cdot F(\lambda_1, K_1, f_1)$; i.e. $\exists F(1, K, f) \in \mathbb{F}_1$ such that $F(\lambda_2, K_2, f_2) = F(1, K, f) \cdot F(\lambda_1, K_1, f_1)$.

So we have $K_2 = K_1 f + K, \lambda_1 = \lambda_2, f_2 = f f_1$.

Let us describe the decomposition of the group $(\mathbb{F}, \cdot)$ determined by its subgroup $(\mathbb{F}_2, \cdot)$. As it was mentioned above the subgroup is normal, so the left and right decompositions are equal.

$$\mathbb{F}/_R\mathbb{F}_2 = \mathbb{F}/_L\mathbb{F}_2 = \{\mathbb{F}_2 \cdot F(\lambda, K, f) = F(\lambda, K, f) \cdot \mathbb{F}_2;\, F(\lambda, K, f) \in \mathbb{F}\}.$$

Here

$$\boldsymbol{E_2} = \{[F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2)] \in \mathbb{F} \times \mathbb{F};\, F^{-1}(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2) \in \mathbb{F}_2, \\ F(\lambda_2, K_2, f_2) \cdot F^{-1}(\lambda_1, K_1, f_1) \in \mathbb{F}_2\}$$

is corresponding equivalence.

Evidently, we have $F(\lambda_1, K_1, f_1)\, \boldsymbol{E_2}\, F(\lambda_2, K_2, f_2)$ if and only if $F(\lambda_2, K_2, f_2) \in \mathbb{F}_2 \cdot F(\lambda_1, K_1, f_1) = F(\lambda\lambda_1, K_1 + K, f_1)$ for suitable $F(\lambda, K, 1) \in \mathbb{F}_2$, i.e. $\exists F(\lambda, K, 1) \in \mathbb{F}_2$ such that $K_2 = K_1 + K, \lambda\lambda_1 = \lambda_2, f_2 = f_1$.

The above obtained decompositions $\mathbb{G}/_R\mathbb{G}_0$, $\mathbb{G}/_L\mathbb{G}_0$, $\mathbb{F}/_R\mathbb{F}_1 = \mathbb{F}/_L\mathbb{F}_1$, $\mathbb{F}/_R\mathbb{F}_2 = \mathbb{F}/_L\mathbb{F}_2$ make possible to construct certain hypergroups.

Theorem 2. The pairs $(\mathbb{F}, \star_i)$ for $i = 1, 2$ and $(\mathbb{G}, \star)$, where

$$x \star_i y = \mathscr{X} \cup \mathscr{Y} \text{ for } x \in \mathscr{X},\, y \in \mathscr{Y},\, \mathscr{X}, \mathscr{Y} \in \mathbb{F}/\mathbb{F}_i,$$
$$x \star y = \mathscr{X} \cup \mathscr{Y} \text{ for } x \in \mathscr{X},\, y \in \mathscr{Y},\, \mathscr{X}, \mathscr{Y} \in \mathbb{G}/_L\mathbb{G}_0$$

are join spaces.

Proof. Considering Lemma 3 the proof is evident. □

We have $F(\lambda, K, f) = F(\mu, P, g) \Leftrightarrow \forall\varphi \in C(J)\colon F(\lambda, K, f)(\varphi) = F(\mu, P, g)(\varphi) \Leftrightarrow \lambda\int_a^b K(x,s)\varphi(s)\mathrm{d}s + f(x) = \mu\int_a^b P(x,s)\varphi(s)\mathrm{d}s + g(x)$.
1. For $\varphi \equiv 0$ we obtain $f(x) \equiv g(x)$.

Denote $\frac{\lambda}{\mu} = \kappa \neq 0$, then $\int_a^b \big(\kappa K(x,s) - P(x,s)\big)\varphi(s)\mathrm{d}s = 0$.
For any $\varphi \in C(J)\colon \int_a^b \big(\kappa K(x,s) - P(x,s)\big)\varphi(s)\mathrm{d}s = 0$. Let $x_0 \in J$ be an arbitrary but fixed chosen. Then $\int_a^b \big(\kappa K(x_0,s) - P(x_0,s)\big)\varphi(s)\mathrm{d}s = 0$, denote $\xi_{x_0}(s) = \kappa K(x_0,s) - P(x_0,s)$, i.e. $\int_a^b \xi_{x_0}(s)\varphi(s)\mathrm{d}s = 0$ for any $\varphi \in C(J)$. This implies that $\xi_{x_0}(s) = 0$ for all $s \in J$, i.e. $\lambda K(x_0,s) = \mu P(x_0,s)$ for any $s \in J$. Since $x_0 \in J$ was an arbitrary point we get $\lambda K(x,s) = \mu P(x,s)$ on $J \times J$. Consequently we proved the implication $F(\lambda, K, f) = F(\mu, P, g)$ implies $f = g$ and $\lambda K = \mu P$.

Define the following equivalence ρ on $\mathbb{F}$ in this way: For $F(\lambda, K, f), F(\mu, P, g) \in \mathbb{F}$ we get $F(\lambda, K, f)\rho F(\mu, P, g)$ whenever $f = g$ and $\lambda K = \mu P$. Evidently ρ is an equivalence relation on $\mathbb{F}$.

We will show that this equivalence ρ is a congruence on the group $(\mathbb{G}, \odot)$. Let $F(\lambda_i, K_i, f_i), F(\lambda, K, f) \in \mathbb{G}$ (for i=1,2) be an arbitrary triple of integral operators, such that the relation $F(\lambda_1, K_1, f_1)\,\rho\,F(\lambda_2, K_2, f_2)$ holds. Then $\lambda_1 K_1 = \lambda_2 K_2$, $f_1 = f_2$ and we have

$$F(\lambda, K, f) \odot F(\lambda_1, K_1, f_1) = F(\lambda\lambda_1, KK_1, \lambda\hat{K} f_1 + f),$$
$$F(\lambda, K, f) \odot F(\lambda_2, K_2, f_2) = F(\lambda\lambda_2, KK_2, \lambda\hat{K} f_2 + f).$$

Now $\lambda\lambda_1 KK_1 = \lambda K\lambda_1 K_1 = \lambda K\lambda_2 K_2$ and $\lambda\hat{K} f_1 + f = \lambda\hat{K} f_2 + f$, thus

$$\big(F(\lambda, K, f) \odot F(\lambda_1, K_1, f_1)\big)\rho\big(F(\lambda, K, f) \odot F(\lambda_2, K_2, f_2)\big).$$

Similarly we obtain that

$$\big(F(\lambda_1, K_1, f_1) \odot F(\lambda, K, f)\big)\rho\big(F(\lambda_2, K_2, f_2) \odot F(\lambda, K, f)\big).$$

So ρ is a congruence.

Consider the ρ–class containing the unit $F(1,1,0)$, i.e. $\rho\big(F(1,1,0)\big) \in \mathbb{F}/\rho$. Suppose $F(\lambda, K, f) \in \rho\big(F(1,1,0)\big)$, i.e.
$F(\lambda, K, f)\rho F(1,1,0)$, i.e. $\lambda K = 1$, $f \equiv 0$. Then

$$F(\lambda, K, f)(\varphi) = \int_a^b \varphi(s)\mathrm{d}s = F(1,1,0)(\varphi)$$

for any $\varphi \in C(J)$. This means that any ρ–class is formed exactly by one integral operator. Choosing one representative of each class we can deduce from $F(\lambda, K, f) = F(\mu, P, g)$ that $\lambda = \mu$, $K = P$, $f \equiv g$ for the representing operator of the corresponding class of the equivalence defined by $[\lambda, K, f] \sim_\rho [\mu, P, g]$ iff $\lambda K = \mu P$, $f = g$. Thus $\sim_\rho$ is an equivalence on $\Xi_{\mathbb{F}}$, where

$$\Xi_{\mathbb{F}} = \mathbb{R}_+ \times C(J \times J) \times C(J).$$

Let $\boldsymbol{L}$ be the equivalence relation on $\mathbb{G}$ determined by the left decomposition of $\mathbb{G}$ by $\mathbb{G}_0$. That is $\mathbb{G}/_L\mathbb{G}_0 = \{F(\lambda, K, f) \odot \mathbb{G}_0; F(\lambda, K, f) \in \mathbb{G}\} = \mathscr{L}(\mathbb{G})$. We show first which pairs of operators from $\mathbb{G}$ belong to the same block of decomposition $\mathscr{L}(\mathbb{G})$. So let $\mathscr{X} \in \mathscr{L}(\mathbb{G})$ be an arbitrary block, $F(\lambda, K, f), F(\mu, P, g) \in \mathscr{X}$. Then $F(\lambda, K, f) \in F(\mu, P, g) \odot \mathbb{G}_0$. This is equivalent to existence of a function $U \in C(J \times J)$ such that $F(\lambda, K, f) = F(\mu, P, g) \odot F(1, U, 0) = F(\mu, PU, g)$.

On the other hand there exists a function $V \in C(J \times J)$ such that $F(\mu, P, g) = = F(\lambda, K, f) \odot F(1, V, 0) = F(\lambda, KV, f)$. From these equalities there follows that $F(\lambda, K, f), F(\mu, P, g)$ belongs to the same block of $\mathscr{L}(G)$, i.e.

$F(\lambda, K, f)\, \boldsymbol{L}\, F(\mu, P, g)$ if and only if $\lambda = \mu$, $f = g$.

We are going to define a binary hyperoperation on the factor set $\mathscr{L}(\mathbb{G}) = \mathbb{G}/_L\mathbb{G}_0$. Remember that for arbitrary pairs $\mathscr{X}, \mathscr{Y} \in \mathscr{L}(\mathbb{G})$ we put

$$\begin{aligned}\mathscr{X} \odot \mathscr{Y} =\{&F(\lambda, K, f);\ F(\lambda, K, f) = F(\lambda_1, K_1, f_1) \odot F(\lambda_2, K_2, f_2),\\ &F(\lambda_1, K_1, f_1) \in \mathscr{X}, F(\lambda_2, K_2, f_2) \in \mathscr{Y}\}.\end{aligned}$$

Then for an arbitrary pair $\mathscr{X}, \mathscr{Y} \in \mathscr{L}(\mathbb{G})$ we define

$$\mathscr{X} \star \mathscr{Y} = \{\mathscr{Z} \in \mathscr{L}(\mathbb{G}), \mathscr{Z} \cap (\mathscr{X} \odot \mathscr{Y}) \neq \emptyset\}.$$

This is the usual construction in algebraic hyperstructure theory.
In what follows we suppose

$$\mathbb{G} = \{F(\lambda, K, f),\ \text{where } \lambda \in \mathbb{R}, \lambda > 0, K \in C_+(J \times J)\}.$$

Proposition 3. The hypergroupoid $(\mathscr{L}(\mathbb{G}), \star)$ is a quasi–hypergroup with the following properties.

1. For any $\mathscr{X} \in \mathscr{L}(\mathbb{G})$ there holds $\mathscr{X} \star \mathbb{G}_0 = \{\mathscr{X}\}$, i.e. $\mathbb{G}_0$ is the right unit of $(\mathscr{L}(\mathbb{G}), \star)$ and $\mathscr{X} \in \mathbb{G}_0 \star \mathscr{X}$.
2. If $\mathscr{X}, \mathscr{Y} \in \mathscr{L}(\mathbb{G})$ are blocks with representing operators $F(\lambda, K_0, f) \in \mathscr{X}$, $F(\mu, P_0, g) \in \mathscr{Y}$ such that $g \in C_+(J)$, $f \in C_+(J)$ we have

$$\mathscr{X} \star \mathscr{Y} = \Big\{\big[F(\lambda\mu, KP, h)\big]; K, P \in C_+(J \times J), h \in C(J), h > f\Big\}.$$

Here by $[\cdot]$ we mean the block with representing operator $\cdot$.

Proof. Let $\mathscr{X} \in \mathscr{L}(\mathbb{G})$ be an arbitrary block and $F(\lambda, K, f) \in \mathscr{X}$ be its arbitrary representing operator. For arbitrary $F(1, P, 0) \in \mathbb{G}_0$ we have

$$F(\lambda, K, f) \odot F(1, P, 0) = F(\lambda, KP, f) \in \mathscr{X},$$

thus $\mathscr{X} \odot \mathbb{G}_0 \subset \mathscr{X}$. Then $\mathscr{X} \star \mathbb{G}_0 = \{\mathscr{Z}, \mathscr{Z} \in \mathscr{L}(\mathbb{G}), \mathscr{Z} \cap (\mathscr{X} \odot \mathbb{G}_0) \neq \emptyset\} = \{\mathscr{X}\}$. Moreover $F(1, P, 0) \odot F(\lambda, K, f) = F(\lambda, KP, \hat{P} f)$. Since any function $U \in C_+(J \times J)$ can be represented in the form $U = KP$ with the fixed function K given above, we have the set $\mathbb{G}_0 \odot \mathscr{X}$ is saturated in the factor set $\mathbb{G}/_L\mathbb{G}_0 = \mathscr{L}(\mathbb{G})$, i.e. $\mathbb{G}_0 \odot \mathscr{X}$ is a union of some blocks of $\mathbb{G}/\mathbb{G}_0$. For $P = 1$ we get $F(\lambda, K, f) \in \mathbb{G}_0 \odot \mathscr{X}$, consequently $\mathscr{X} \in \mathbb{G}_0 \star \mathscr{X}$.

Let us denote $\Phi = \{[F(\lambda\mu, KP, h)]; K, P \in C_+(J \times J), h \in C(J), h > f\}$. Let $[F(\xi, U, \varphi)] \in \mathscr{X} \star \mathscr{Y}$. Then $\xi = \lambda\mu$, $U = KP$ for a suitable pair $K, P \in C_+(J \times J)$ and $\varphi = \lambda\hat{K} g + f > f$ since $\lambda\hat{K} g > 0$ on J, which implies $\mathscr{X} \star \mathscr{Y} \subset \Phi$.

Suppose $F(\xi, U, \varphi) \in \Phi$. Then $\xi = \lambda\mu$ and there exists $K_1, P_1 \in C_+(J \times J)$ such that $U = K_1 P_1$, moreover φ is a continuous function on J, $f(x) < \varphi(x)$ for any $x \in J$.

Consider a function $K\colon J \times J \to \mathbb{R}$ defined in this way: Denote

$$\psi(x) = \frac{\varphi(x) - f(x)}{g(x)} \qquad \text{for } x \in J.$$

Since g is a positive function, ψ is well defined and it is also positive and continuous on the segment J. That is $\psi \in C_+(J)$. Define

$$K(x, y) = \begin{cases} \frac{1}{\lambda}\,\psi(x) & \text{if } x = y \in J, \\ \frac{1}{\lambda}\,\psi(y) & \text{if } x \neq y,\ y \in J. \end{cases}$$

In detail $K(x_1, y) = K(x_2, y)$ for all pairs $x_1, x_2 \in J$, if $x_1 \neq y \neq x_2$, i.e. the function $K(x, y)$ on the square $J \times J$ is continuously extended from the diagonal $J \times J$ in such a way that this function $K(x, y)$ on segments $y = y_0$, $x = t$, $t \in J$ is constant. Thus $K \in C_+(J \times J)$ and $F(\lambda, K, f)\boldsymbol{L}F(\lambda, K_0, f)$, i.e. $F(\lambda, K, f) \in \mathscr{X}$. Define a function $P\colon J \times J \to \mathbb{R}$ by the equality $P_2 = \frac{1}{K} K_1 P_1 = \frac{U}{K}$. Then $F(\mu, P_2, g) \in \mathscr{Y}$ and we have

$$\begin{aligned} &[F(\xi, U, \varphi)] = [F(\lambda\mu, KP_2, \psi g + f)] = \\ &\qquad [F(\lambda\mu, KP_2, \lambda\hat{K} g + f)] = [F(\lambda, K, f) \odot F(\mu, P_2, g)] \in \mathscr{X} \star \mathscr{Y}, \end{aligned}$$

thus $\Phi \subset \mathscr{X} \star \mathscr{Y}$. Consequently the equality $\mathscr{X} \star \mathscr{Y} = \Phi$ holds. □

The description of the structure of $\mathbb{G}/_R\mathbb{G}_0$ in detail, i.e. dual hyperstructure on the carrier $\mathbb{G}/_R\mathbb{G}_0$, seems to be open.

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