Šárka HOŠKOVÁ
Jan CHVALINA
Pavlína RAČKOVÁ

HYPERGROUPS OF INTEGRAL OPERATORS IN CONNECTIONS WITH TRANSFORMATION STRUCTURES

Reviewer: Jiří MOUČKA

Abstract:

This contribution is a continuation of the paper [4] (here not defined concepts can be found there). It uses the results from it and the fact that the basic group of integral operators contains an invariant subgroup, which allows us to obtain a closed, invertible, reflexive and normal subhypergroup of the target transposition hypergroup. We will construct a discrete transformation hypergroup—in fact an action of hypergroup of integral operators on the space of continuous functions, which are created by Fredholm integral equations of the second kind, as a phase set.

1 Preliminaries

Recall first the basic terms and definitions. A *hypergroupoid* is a pair $(H, \cdot)$ where H is a (nonempty) set and $\cdot : H \times H \to \mathscr{P}^*(H)$ $(= \mathscr{P}(H) \setminus \{\emptyset\})$ is a binary hyperoperation on the set H. If $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in H$, (associativity), then $(H, \cdot)$ is called a *semihypergroup*. A semihypergroup $(H, \cdot)$ is said to be a *hypergroup* if the following axiom $a \cdot H = H = H \cdot a$ for all a $\in H$, (the reproduction axiom), is satisfied. Here, for $A, B \subseteq H$, $A \neq \emptyset \neq B$ we define as usual $A \cdot B = \bigcup\{a \cdot b;\, a \in A, b \in B\}$. A commutative hypergroup $(H, \cdot)$ satisfying the so called *transposition axiom*, i.e., for any quadruple $a, b, c, d \in H$ such that $a/b \cap c/d \neq 0$ we have $a \cdot d \cap b \cdot c \neq 0$, where $a/b = \{x \in H;\, a \in x \cdot b\}$ is called a *join space*, (see e.g. [2], [3], [12]).

2 Further properties of join spaces of operators

Let us return to the hypergroup constructed in Chapter 3 of [4].

Denote $\mathbb{F} = \{F(\lambda, K, f) :\ K(x, s) \in C(J \times J),\ f \in C_+(J),\ \lambda \neq 0\}$, where $F(\lambda, K, f)$ is given by

$$F(\lambda, K, f)(\varphi(x)) = \lambda \int_a^b K(x, s)\varphi(s)\,\mathrm{d}s + f(x). \tag{1}$$

In the sequel the operation " $\cdot$ " means

$$F(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2) = F(\lambda_1\lambda_2, K_2 f_1 + K_1, f_1 f_2) \tag{2}$$

(see [4]). In Proposition 1 of [4] it was shown that $\mathbb{F}$ with the relation "$\leqq$" defined by

$$F(\lambda_1, K_1, f_1) \leqq F(\lambda_2, K_2, f_2)$$

$$\text{if and only if } \lambda_1 = \lambda_2,\ f_1(x) \equiv f_2(x) \text{ and } K_1(x, s) \leqq K_2(x, s)$$

is a noncommutative ordered group.

Further the hyperoperation "$*$" is defined as a union of upper ends:

$$\begin{aligned} F(\lambda_1,K_1, f_1) * F(\lambda_2, K_2, f_2) = & \\ &= \{F(\lambda, K, f) \in \mathbb{F};\ F(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2) \leqq F(\lambda, K, f)\} \\ &= \{F(\lambda, K, f) \in \mathbb{F};\ F(\lambda_1\lambda_2, K_2 f_1 + K_1, f_1 f_2) \leqq F(\lambda, K, f)\} \\ &= \{F(\lambda_1\lambda_2, K, f_1 f_2);\ K_2(x, s) f_1(x) + K_1(x, s) \leqq K(x, s), [x, s] \in J \times J\}. \end{aligned} \tag{3}$$

It was proved that $(\mathbb{F}, *)$ is a join space, see Theorem 1 of [4].

The just defined binary hyperoperation determines ternary relation $\mathbb{B}$ (to lie between) on the set $\mathbb{F}$ in such a way that

$$\big[F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2), F(\lambda_3, K_3, f_3)\big] \in \mathbb{B} \text{ whenever } F(\lambda_2, K_2, f_2) \in \big[F(\lambda_1, K_1, f_1) * F(\lambda_3, K_3, f_3).$$

Evidently this relation is not symmetrical, i.e., the implication

$$\big[F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2), F(\lambda_3, K_3, f_3)\big] \in \mathbb{B} \Rightarrow \big[F(\lambda_3, K_3, f_3), F(\lambda_2, K_2, f_2), F(\lambda_1, K_1, f_1)\big] \in \mathbb{B}$$

is not valid.

Other properties of this relation seem not to be established up to now.

Definition 1. [11], [12] A subhypergroup $(S, \bullet)$ of a hypergroup $(H, \bullet)$ is called
closed if $a/b \subseteq S$ and $b\backslash a \subseteq S$ for all $a, b \in S$,
invertible if $a/b \approx S$ implies $b/a \approx S$, and $b\backslash a \approx S$ implies $a\backslash b \approx S$ for all $a, b \in H$,
reflexive if $a\backslash S = S/a$ for all $a \in H$,
normal if $a \bullet S = S \bullet a$ for all $a \in H$.

Note that the notion of being a subhypergroup, and each of the properties for a subhypergroup in the definition above is self-dual. In any hypergroup, and invertible subhypergroup is closed. In a transposition hypergroup, a closed and normal subhypergroup is reflexive. Thus in a transposition hypergroup, an invertible and normal subhypergroup is closed and reflexive. A normal invertible subhypergroup can be considered to be a hypergroup-analogy of a normal subgroup of a group. For a transposition hypergroup, a reflexive closed subhypergroup plays an analogous role.

The following concepts were defined by James Jantosciak in [11, p. 80].

Theorem 1. Let $J \subseteq \mathbb{R}$ and $\mathbb{F}_1 = \{F(1, K, f) : K \in C(J \times J), f \in C_+(J)\}$. Let the hyperoperation "$*$" be define by (3). The subhypergroupoid $(\mathbb{F}_1, *)$ of the join space $(\mathbb{F}, *)$ is a subhypergroup of $(\mathbb{F}, *)$ and it is closed, invertible, reflexive and normal.

Proof. The subhypergroupoid $(\mathbb{F}_1, *)$ is a subhypergroup when it satisfies axiom of reproduction, i.e. $F * \mathbb{F}_1 = \mathbb{F}_1 * F = \mathbb{F}_1$ for all $F = F(1, K, f) \in \mathbb{F}_1$.
a) $F(1, K, f) * \mathbb{F}_1 \subset \mathbb{F}_1$ is evident.
b) To prove $\mathbb{F}_1 \subset F(1, K, f) * \mathbb{F}_1$ let us recall that

$$F(1, K, f) * F_2(1, K_2, f_2) = \bigcup_{K_2, f_2} \{F(1, K_L, f_2 f) : K_2 f + K \leq K_L\}.$$

For any $F_3(1, K_3, f_3) \in \mathbb{F}_1$ let us choose $f_2 = \frac{f_3}{f}$ and $K_2 \leqq \frac{K_3 - K}{f}$. Then evidently $F_3 \in F * F_2 \subset F * \mathbb{F}_1$. The second equality $\mathbb{F}_1 * F = \mathbb{F}_1$ can be proved analogously.

1. Closeness of $(\mathbb{F}_1, *)$: For any pair of operators $F_1(1, K_1, f_1), F_2(1, K_2, f_2) \in \mathbb{F}_1$ we have by Lemma 4 from [4]

$$F_1(1, K_1, f_1)/F_2(1, K_2, f_2) = \left\{F\left(1, K, \frac{f_1}{f_2}\right) : K \leqq K_1 - K_2 \frac{f_1}{f_2}\right\} \in \mathbb{F}_1,$$

$$F_2(1, K_2, f_2)\backslash F_1(1, K_1, f_1) = \left\{F\left(1, K, \frac{f_1}{f_2}\right) : K \leqq \frac{K_1 - K_2}{f_2}\right\} \in \mathbb{F}_1.$$

   It holds for all operators from $\mathbb{F}_1$, i.e., the subhypergroup $(\mathbb{F}_1, *)$ is closed.

2. Invertibility of $(\mathbb{F}_1, *)$: Suppose that $F_1(\lambda_1, K_1, f_1), F_2(\lambda_2, K_2, f_2) \in \mathbb{F}$ are the operators satisfying

$$F_1(\lambda_1, K_1, f_1)/F_2(\lambda_2, K_2, f_2) \approx \mathbb{F}_1.$$

   By Lemma 4 from [4]

$$F_1(\lambda_1, K_1, f_1)/F_2(\lambda_2, K_2, f_2) = \left\{F\left(1, K, \frac{f_1}{f_2}\right) : K \leqq K_1 - K_2 \frac{f_1}{f_2}\right\} \in \mathbb{F}_1,$$

   i.e., $\frac{\lambda_1}{\lambda_2} \equiv 1$, which means $\lambda_1 = \lambda_2$. Then

$$F_2(\lambda_2, K_2, f_2)/F_1(\lambda_1, K_1, f_1) = \left\{F\left(1, K, \frac{f_2}{f_1}\right) : K \leqq K_2 - K_1 \frac{f_2}{f_1}\right\} \in \mathbb{F}_1.$$

   Similarly $F_2(\lambda_2, K_2, f_2)\backslash F_1(\lambda_1, K_1, f_1) \approx \mathbb{F}_1$,

$$F_2(\lambda_2, K_2, f_2)\backslash F_1(\lambda_1, K_1, f_1) = \left\{F\left(1, K, \frac{f_1}{f_2}\right) : K \leqq \frac{K_1 - K_2}{f_2}\right\} \in \mathbb{F}_1,$$

   i.e., $\lambda_1 = \lambda_2$. Further,

$$F_1(\lambda_1, K_1, f_1)\backslash F_2(\lambda_2, K_2, f_2) = \left\{F\left(1, K, \frac{f_2}{f_1}\right) : K \leqq \frac{K_2 - K_1}{f_1}\right\} \in \mathbb{F}_1.$$

   Thus, $(\mathbb{F}_1, *)$ is an invertible subhypergroup of the hypergroup $(\mathbb{F}, *)$.

3. Reflexivity of $(\mathbb{F}_1, *)$: Suppose $F(\lambda, K, f) \in \mathbb{F}$, then

$$F(\lambda, K, f)\backslash\mathbb{F}_1 = \bigcup\{F(\lambda, K, f)\backslash F_2(1, K_2, f_2) : F_2(1, K_2, f_2) \in \mathbb{F}_1\} = \\ = \bigcup_{f_2}\bigcup_{K_2}\left\{F\left(\frac{1}{\lambda}, K_L, \frac{f_2}{f}\right) : K_L \leqq \frac{K_2 - K}{f}\right\}$$

and similarly

$$\mathbb{F}_1/F(\lambda, K, f) = \bigcup\{F_3(1, K_3, f_3)/F(\lambda, K, f) : F_3(1, K_3, f_3) \in \mathbb{F}_1\} = \\ = \bigcup_{f_3}\bigcup_{K_3}\left\{F\left(\frac{1}{\lambda}, K_{LL}, \frac{f_3}{f}\right) : K_{LL} \leqq K_3 - K\,\frac{f_3}{f}\right\},$$

for $f_3 = f_2$ evidently

$$\bigcup_{f_2}\bigcup_{K_2}\left\{F\left(\frac{1}{\lambda}, K_L, \frac{f_2}{f}\right) : K_L \leqq \frac{K_2 - K}{f}\right\} = \bigcup_{f_3}\bigcup_{K_3}\left\{F\left(\frac{1}{\lambda}, K_{LL}, \frac{f_3}{f}\right) : K_{LL} \leqq K_3 - K\,\frac{f_3}{f}\right\}$$

and we obtain, that $(\mathbb{F}_1, *)$ is reflexive.

4. Normality of $(\mathbb{F}_1, *)$: Suppose $F(\lambda, K, f) \in \mathbb{F}$, then

$$F(\lambda, K, f) * \mathbb{F}_1 = \bigcup\{F(\lambda, K, f) * F_2(1, K_2, f_2); F_2(1, K_2, f_2) \in \mathbb{F}_1\} = \\ = \bigcup_{f_2}\bigcup_{K_2}\{F(\lambda, K_L, ff_2) : K_L \geqq K_2 f + K\}$$

and similarly

$$\mathbb{F}_1 * F(\lambda, K, f) = \bigcup_{f_3}\bigcup_{K_3}\{F(\lambda, K_{LL}, f_3 f) : K_{LL} \geqq K f_3 + K_3\},$$

for $f_2 = f_3$ evidently

$$\bigcup_{f_2}\bigcup_{K_2}\{F(\lambda, K_L, ff_2) : K_L \geqq K_2 f + K\} = \bigcup_{f_3}\bigcup_{K_3}\{F(\lambda, K_{LL}, f_3 f) : K_{LL} \geqq K f_3 + K_3\}$$

and

$$F(\lambda, K, f) * \mathbb{F}_1 = \mathbb{F}_1 * F(\lambda, K, f)$$

holds for any operator $F(\lambda, K, f) \in \mathbb{F}$, hence $(\mathbb{F}_1, *)$ is normal. □

We have verified properties of the corresponding subhypergroups directly. Nevertheless, invertibility and normality imply closeness and reflexivity—by [12] or [11, p. 80]. Thus it suffices to prove the first and the second property of the subhypergroup $(\mathbb{G}, *)$.

In a similar way we get that the subset of $(\mathbb{F}, *)$ formed by integral operators of the form $F(1, K, 1)$, which means operators $F(\lambda, K, f)$ where $\lambda = 1,\ f \equiv 1$, is also a carrier of a subhypergroup of $(\mathbb{F}, *)$ which is closed, invertible, reflexive and normal.

It is to be noted that in the case of Volterra integral operators of the convolution type in [10] there was constructed an inclusion embedding (i.e., an inclusive injective homomorphism) of the semihypergroup formed by operators commuting with given Volterra operator into the centralizer hypergroup of certain transformation of half plain complex numbers.

Recall that by Volterra operator of the convolution type (see e.g. [14]) we mean operator of the form

$$V(\lambda, k, f)(\varphi) = \lambda \int_0^x k(x-s)\varphi(s)ds + f(x) \tag{4}$$

where all continuous functions φ are of exponential order.

The mentioned embedding is carried by classical Laplace transform of the form

$$L(\varphi) = \int_0^\infty \mathrm{e}^{-pt}\varphi(t)\mathrm{d}t,$$

where k, f, φ are continuous functions, where φ is of the bounded exponential growth on $J = \langle 0, \infty)$.

This construction is based on Theorem 1 proved in [10]. If the improper integral $\int_0^\infty \mathrm{e}^{-pt}k(t) * \varphi(t)\mathrm{d}t$ absolutely converges, then applying the Laplace transform to the convolution of functions k, φ we obtain (with respect to the product theorem)

$$L(k(t) * \varphi(t)) = \int_0^\infty \mathrm{e}^{-pt} \int_0^t k(t,s)\varphi(s)\mathrm{d}s\mathrm{d}t = K(p)\Phi(p).$$

Consequently, if using the Laplace transform $L(f(t)) = F(P)$, $L(\varphi(t)) = \Phi(p)$ and $L(k(t)) = K(p)$. Then we obtain

$$L(V(\lambda, k, f)(\varphi)) = \lambda K(p)\Phi(p) + F(p).$$

Consider the half-plane of complex numbers $\Omega = \{z; Rez > 0\}$. For $\lambda \in \mathbb{R}^+$, $K, F \in \mathbb{C}(\Omega)$, $F(p)$ is different from 0 for any $p \in \Omega$ we define $T(\lambda, K, F)\Phi = = \lambda K(p)\Phi(p) + F(p)$, $p \in \Omega$, $\Phi \in \mathbb{C}(\Omega)$. On the set $\mathscr{T}(\Omega)$ of such operators we consider the binary operation

$$\begin{aligned}T(\lambda, K, F).T(\mu, S, G)\Phi(p) = T(\lambda\mu, \lambda S + K, FG)\Phi = \\ = \lambda^2\mu S(p)\Phi(p) + \lambda\mu K(p)\Phi(p) + F(p)G(p) = \\ = T(\lambda\mu, K, FG)\Phi(p) + T(\lambda^2\mu, S, 0)\Phi(p),\end{aligned}$$

thus $T(\lambda, K, F).T(\mu, S, G) = T(\lambda\mu, \lambda S + K, FG)$.

This is a certain generalization of the concept of translation operators, which are investigated in paper [8]. In a similar way as in that paper, it is easy to show in our case as well that our groupoid $(\mathscr{T}(\Omega), .)$ is a noncommutative group. Indeed, in the same way as in the case of $(\mathscr{V}_c(J), .)$, the just defined operation is associative; the operator $T(1, 0, 1)$, i.e., if $\lambda = 1$, $K(p) \equiv 0$, $F(p) \equiv 1$ is the unit, i.e., $T(1, 0, 1)\Phi(p) \equiv 1$ and $T(1, 0, 1).T(\lambda, K, F) = T(\lambda, K, F).T(1, 0, 1) = = T(\lambda, K, F)$. For arbitrary $T(\lambda, K, F) \in \mathscr{T}(\Omega)$ its inverse operator is

$$T^{-1}(\lambda, K, F) = T(\frac{1}{\lambda}, -\frac{K}{\lambda}, \frac{1}{F}).$$

Moreover, with the use of suitable ordering on $\mathscr{T}(\Omega)$ we obtain a transposition hypergroup in a similar way as in paper [8]. Define $\mathbf{L}(V(\lambda, k, f)) = T(\lambda, K, F)$ if $L(V(\varphi)) = T(\phi)$.

The following theorem proven in [10] allows us to construct an embedding of the centralizer semihypergroup of Volterra convolution operators into the centralizer hypergroup of the above transformation operators $T(\lambda, K, F)$ of the half-plane Ω.

Theorem 2. The Laplace transformation defined on the set of Volterra operators $\mathscr{V}_c(J)$ of the convolution type is an embedding (i.e., an injective homomorphism) $\mathbf{L}$ of the semigroup $(\mathscr{V}_c(J), .)$ into the group $(\mathscr{T}(\Omega), .)$.

In connection with the above considerations and Generalized Product Theorem (Efros-theorem), see e.g. [13], the following problem arises: If Laplace transformation of a function $f(t)$ is equal to $F(p)$, i.e., $L\big(f(t)\big) = F(p)$ and $L\big(\varphi, (t, \tau)\big) = \Phi(p)\mathrm{e}^{\tau q(p)}$, where $\Phi(p)$ and $q(p)$ are analytical functions, then

$$L\left(\int_o^\infty f(\tau)\varphi(t, \tau)\mathrm{d}\tau\right) = \Psi(p)F\big(q(p)\big).$$

The open problem is whether similarly as in the case of Volterra integral operators the Laplace transform or some of the other classical transform carrying algebraical embedding a join space into suitable other hyperstructure.

3 Centralizer transformation semihypergroup of integral operators

In this section we will construct a discrete transformation hypergroup—in fact an action of hypergroup of integral operators on the space of continuous functions as a phase set. To achieve this we create a certain modification of Example 1 from [9].

Let us recall (compare Proposition 1 of [4]) that $(\mathbb{F}, \cdot)$ is the group of Fredholm integral operators of the second kind given by (2).

Denote by $D^{f_1,f_2}_{K_1,K_2}(x,s)$ the following determinant:

$$D^{f_1,f_2}_{K_1,K_2}(x,s) = \begin{vmatrix} f_1(x) & f_2(x) \\ K_1(x,s) & K_2(x,s) \end{vmatrix}.$$

Lemma 1. *Fredholm integral operators $F(\lambda_1, K_1, f_1)$, $F(\lambda_2, K_2, f_2)$ are commuting in the group $(\mathbb{F}, \cdot)$ if and only if*

$$D^{f_1,f_2}_{K_1,K_2}(x,s) = K_2(x,s) - K_1(x,s)$$

for all points $[x,s] \in \langle a,b\rangle \times \langle a,b\rangle$.

Proof. Let $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2) \in \mathbb{F}$ be commuting integral operators, i.e.

$$F(\lambda_1, K_1, f_1) \cdot F(\lambda_2, K_2, f_2) = F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1),$$
$$F(\lambda_1\lambda_2, K_2 f_1 + K_1, f_1 f_2) = F(\lambda_2\lambda_1, K_1 f_2 + K_2, f_2 f_1).$$

Then $K_2 f_1 + K_1 = K_1 f_2 + K_2$, i.e., $f_1 K_2 - f_2 K_1 = K_2 - K_1$, hence $D^{f_1,f_2}_{K_1,K_2}(x,s) = K_2(x,s) - K_1(x,s)$ for all elements $[x,s] \in \langle a,b\rangle \times \langle a,b\rangle$. Evidently the procedure can be reversed. □

Definition 2. Let X be a set, $(G, \bullet)$ be a semihypergroup and $\pi : X \times G \to X$ a mapping such that

$$\pi(\pi(x,t),s) \in \pi(x, t \bullet s), \text{ where } \pi(x, t \bullet s) = \{\pi(x,u); u \in t \bullet s)\} \quad (5)$$

for each $x \in X$, $s,t \in G$. Then (X, G, π) is called a *discrete transformation semihypergroup* or an action of the semihypergroup G on the phase set X. The mapping π is usually said to be simply an action.

More generally, it is possible to consider the situation, where the phase space X is endowed with some additional structure. An interesting case is given in [3], [5].

Remark 1. The condition (5) used above is called Generalized Mixed Associativity Condition, shortly GMAC.

By a *centralizer* of an element a of the group G we mean—as usual—its subgroup $\mathbb{C}_G(a) = \{x \in G; ax = xa\}$. A centralizer of an element $F(\lambda, K, f) \in \mathbb{F}$ is a subgroup

$$\begin{aligned}\mathbb{C}_{\mathbb{F}}\big(F(\lambda, K, f)\big) &= \{F(\mu, L, g) \in \mathbb{F}; F(\lambda, K, f) \cdot F(\mu, L, g)\\ &= F(\mu, L, g) \cdot F(\lambda, K, f)\} =\\ &= \{F(\mu, L, g) \in \mathbb{F}; D^{f,g}_{K,L}(x, s) = L(x, s) - K(x, s)\}\end{aligned}$$

for any pair $[x, s] \in \langle a, b\rangle \times \langle a, b\rangle$.

Definition 3. Let $F(\lambda_0, K_0, f_0) \in \mathbb{F}$ be an arbitrary but fixed operator. Denote by $\mathbb{C}_{\mathbb{F}}(F(\lambda_0, K_0, f_0)) = \mathbb{C}_{\mathbb{F}}$ the centralizer of the operator $F(\lambda_0, K_0, f_0)$ within the group $(\mathbb{F}, \cdot)$. Let us define a hyperoperation $\star\colon \mathbb{C}_{\mathbb{F}} \times \mathbb{C}_{\mathbb{F}} \to \mathbb{P}^*(\mathbb{C}_{\mathbb{F}})$ as follows

$$\begin{aligned}&F(\lambda_1, K_1, f_1) \star F(\lambda_2, K_2, f_2) =\\ &\qquad \{F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1); n \in \mathbb{N}_0\}\end{aligned}$$

for any pair of operators $F(\lambda_1, K_1, f_1), F(\lambda_2, K_2, f_2) \in \mathbb{C}_{\mathbb{F}}(F(\lambda_0, K_0, f_0))$.

Denote $\mathbb{M}\big(F(\lambda_0, K_0, f_0)\big) = \big(C\langle a, b\rangle, (\mathbb{C}_{\mathbb{F}}, \star), \delta\big)$ where the mapping $\delta\colon C\langle a, b\rangle \times \mathbb{C}_{\mathbb{F}} \to C\langle a, b\rangle$ is defined by

$$\begin{aligned}\delta\big(\varphi, F(\lambda, K, f)\big) &= \big(F(\lambda_0, K_0, f_0) \cdot F(\lambda, K, f)\big)(\varphi(x))\\ &= F(\lambda_0\lambda, Kf_0 + K_0, f_0 f)(\varphi(x))\\ &= \lambda_0\lambda \int_a^b \big(K(x, s) f_0(x) + K_0(x, s)\big)\varphi(s)\mathrm{d}s + f_0(x) f(x).\end{aligned}$$

Proposition 1. The system $\mathbb{M}\big(F(\lambda_0, K_0, f_0)\big) = \big(C\langle a, b\rangle, (\mathbb{C}_{\mathbb{F}}, \star), \delta\big)$ is a discrete transformation semihypergroup with the phase set $C\langle a, b\rangle$ and the phase semihypergroup $(\mathbb{C}_{\mathbb{F}}, \star)$.

Proof. We show first, that $(\mathbb{C}_{\mathbb{F}}, \star)$ is a semihypergroup.

Considering the binary relation $\boldsymbol{r} \subset \mathbb{C}_{\mathbb{F}} \times \mathbb{C}_{\mathbb{F}}$ defined by $F(\lambda_1, K_1, f_1)\, \boldsymbol{r}\, F(\lambda_2, K_2, f_2)$ if and only if $F(\lambda_2, K_2, f_2) = F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_1, K_1, f_1)$ for some $n \in \mathbb{N}_0$, we get without an effort that $(\mathbb{C}_{\mathbb{F}}, \boldsymbol{r})$ is a quasi-ordered monoid. Evidently, the relation $\boldsymbol{r}$ is a quasi-ordering on $\mathbb{C}_{\mathbb{F}}$.

Further, for any triple $F(\lambda_1, K_1, f_1)$, $F(\lambda_2, K_2, f_2)$, $F(\lambda_3, K_3, f_3) \in \mathbb{C}_{\mathbb{F}}$ such that $F(\lambda_1, K_1, f_1)\, \boldsymbol{r}\, F(\lambda_2, K_2, f_2)$, i.e.,
$F(\lambda_2, K_2, f_2) = F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_1, K_1, f_1)$ for a suitable $n \in \mathbb{N}_0$, we have $F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_1, K_1, f_1) \cdot F(\lambda_3, K_3, f_3) = F(\lambda_2, K_2, f_2) \cdot F(\lambda_3, K_3, f_3)$ which means $(F(\lambda_1, K_1, f_1) \cdot F(\lambda_3, K_3, f_3))\, \boldsymbol{r}\, (F(\lambda_2, K_2, f_2) \cdot F(\lambda_3, K_3, f_3))$. Similarly
$F(\lambda_3, K_3, f_3) \cdot F(\lambda_2, K_2, f_2) = F(\lambda_3, K_3, f_3) \cdot F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_1, K_1, f_1)$, i.e., $(F(\lambda_3, K_3, f_3) \cdot F(\lambda_1, K_1, f_1))\, \boldsymbol{r}\, (F(\lambda_3, K_3, f_3) \cdot F(\lambda_2, K_2, f_2))$, therefore $(\mathbb{C}_{\mathbb{F}}, \cdot, \boldsymbol{r})$ is a quasi-ordered monoid. Now defining a binary hyperoperation "$\star$" by

$$F(\lambda_1, K_1, f_1) \star F(\lambda_2, K_2, f_2) = \{F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1); n \in \mathbb{N}_0\}$$

we get

$$F(\lambda_1, K_1, f_1) \star F(\lambda_2, K_2, f_2) = \boldsymbol{r}\big(F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1)\big) = \big[F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1)\big)_{\boldsymbol{r}}$$

and by [6, p. 146] Theorem 1.3., or [8] Proposition 1 we obtain that $(\mathbb{C}_{\mathbb{F}}, \star)$ is a semihypergroup (non commutative in general).

It remains to show that GMAC (see (5)) is satisfied. Let $\varphi \in C\langle a, b\rangle$ be an arbitrary function, $F(\lambda_1, K_1, f_1)$, $F(\lambda_2, K_2, f_2) \in \mathbb{C}_{\mathbb{F}}$ be arbitrary operators. We have

$$\begin{aligned}
\delta(\delta(\varphi, F(\lambda_1, K_1, f_1)), F(\lambda_2, K_2, f_2)) &= \\
&= \delta(F(\lambda_0, K_0, f_0) \cdot (F(\lambda_1, K_1, f_1)(\varphi(x)), F(\lambda_2, K_2, f_2)) \\
&= \delta(F(\lambda_0\lambda_1, K_1 f_0 + K_0, f_0 f_1)(\varphi(x)), F(\lambda_2, K_2, f_2)) \\
&= F(\lambda_0, K_0, f_0) \cdot F(\lambda_2, K_2, f_2) \cdot F(\lambda_0\lambda_1, K_1 f_0 + K_0, f_0 f_1)(\varphi(x)) \\
&= F(\lambda_0^2\lambda_1\lambda_2, K_1 f_0^2 f_2 + K_2 f_0^2 + K_0 f_0 + K_0, f_0^2 f_1 f_2)(\varphi(x)).
\end{aligned}$$

On the other hand for

$$F = F(\lambda_0, K_0, f_0) \cdot F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1) = = F(\lambda_0\lambda_1\lambda_2, K_1 f_0 f_2 + K_2 f_0 + K_0, f_0 f_1 f_2)$$

and for an arbitrary function $\varphi = C\langle a, b\rangle$ we have

$$F(\lambda_0^2\lambda_1\lambda_2, K_2 f_0^2 f_1 + K_1 f_0^2 + K_0 f_0 + K_0, f_0^2 f_1 f_2)(\varphi(x)) =$$

$$= F(\lambda_0, K_0, f_0) \cdot F(\lambda_0\lambda_1\lambda_2, K_1 f_0 f_2 + K_2 f_0 + K_0, f_0 f_1 f_2)(\varphi(x)) \in$$
$$\{(F(\lambda_0, K_0, f_0) \cdot F_n)(\varphi(x)); F_n \in F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1)\}$$
$$= \big\{\delta(\varphi, F_n);\ F_n \in \{F^n(\lambda_0, K_0, f_0) \cdot F(\lambda_2, K_2, f_2) \cdot F(\lambda_1, K_1, f_1);\ n \in \mathbb{N}_0\}\big\}$$
$$= \{\delta(\varphi, F_n);\ F_n \in F(\lambda_1, K_1, f_1) \star F(\lambda_2, K_2, f_2)\}$$
$$= \delta\big(\varphi, F(\lambda_1, K_1, f_1) \star F(\lambda_2, K_2, f_2)\big). \qquad \square$$

Note that transformation semihypergroup $\mathbb{M}$ from the above proposition determines in a very natural way two basic topologies on the set of continuous functions $C\langle a, b\rangle$. More precisely, the semihypergroup $\mathbb{M}$ defines mappings $\mathrm{Cl}^+, \mathrm{Cl}^-\colon \mathscr{P}^*(C\langle a, b\rangle) \to \mathscr{P}^*(C\langle a, b\rangle)$ by

$$\mathrm{Cl}^+(\Phi) = \big\{\delta(\varphi, F(\lambda, K, f));\ \varphi \in \Phi, F(\lambda, K, f) \in \mathbb{C}_{\mathbb{F}}\big\}$$

for any $\emptyset \neq \Phi \subset C\langle a, b\rangle$, $\mathrm{Cl}^+(\emptyset) = \emptyset$ and

$$\mathrm{Cl}^-(\Phi) = \Psi,$$
$$\delta(\Psi, \mathbb{C}_{\mathbb{F}}) = \big\{\delta(\varphi, F(\lambda, K, f));\ \varphi \in \Psi, F(\lambda, K, f) \in \mathbb{C}_{\mathbb{F}}\big\} = \Psi,$$

for any $\emptyset \neq \Phi \subset C\langle a, b\rangle$, $\mathrm{Cl}^-(\emptyset) = \emptyset$.

It is possible to verify that Cl^+ and Cl^- are totally additive topological closure operators.

The theory of hyperstructures finds its applications in such technical models where multi-valued mapping occurred, i.e., where the result of some operation or mapping is not a single element but the whole nonempty set of elements. Such models can be encountered in noncommutative algebra, geometrical structures, in physics (e.g. nuclear fission, the interaction between a foton with certain energy and an electron). Another motivation for investigation of hyperstructures yields from technical processes as a time sequence of military car repairs with respect to its roadability consequences and its operational behaviour. Moreover, some ideas leading naturally to multistructures are also coming from quantum mechanics, quantum optics with applications as quantum cryptography or, in particular, development of quantum computers. The basic idea consist in fact that quantum objects can be simulated in more different states simultaneously.

The study of integral operators, from which the appropriate hyperstructures are created, is a part of an integro-differential equations theory. These equations are used for the modelling and solving of electric circuits with small nonlinearities. Such structures occur as well in the building industry during calculations of bridge designs with deflection.

Fredholm integral equations of the first kind, which are studied on spaces of appropriate integral operators, serve as models for measurement in various parts of modern physics, e.g. optics, electricity and magnetism, nucleonics, etc. The kernel of an appropriate integral equation represents a reaction of an apparatus on the measured quantity. The right hand side of this equation represents an interaction of the measured quantity with the apparatus. The aim of the measuring is to set the physical quantity incorporated in a data contained in an integrand of the integral operator. The kernel of the integral equation incorporates as well a characteristic of the used apparatus. Such a procedure has a great usage during the reconstruction of pictures gained from satellites.

The authors thanks to prof. František Cvachovec for his contribution to improvement of this text.

Introduction of Authors:

References

[1] BERÁNEK, J., CHVALINA, J. From groups of linear functions to noncommutative transposition hypergroups. *Dept. Math. Report Series*, No. 7, Univ. of South Bohemia, 1999, p. 1–10.

[2] CORSINI, P., LEOREANU V. *Applications of Hyperstructure Theory*. Dordrecht – Boston – London, Kluwer Academic Publishers 2003.

[3] HOŠKOVÁ, Š., CHVALINA, J. Discrete transformation hypergroups and transformation hypergroups with phase tolerance space. Accepted to *Discrete Mathematics*, 2007.

[4] HOŠKOVÁ, Š., CHVALINA, J., RAČKOVÁ, P. Noncommutative join spaces of integral operators and related hyperstructures. *Advances in MT*, 2006, vol. 1, no. 1, University of Defence, Brno, p. 7–18.

[5] HOŠKOVÁ, Š., CHVALINA, J. Transformation tolerance hypergroup. *Thai Journal of mathematics*, 2006, Vol 4, Nr. 1, p. 63–72.

[6] CHVALINA, J. *Function Graphs, Quasi-ordered Sets and Commutative Hypergroups*. MU Brno 1995, 205 pp. (in Czech)

[7] CHVALINA J., CHVALINOVÁ , L. Join Spaces of Linear Ordinary Differential Operators of the Second Order. In *Coll. on Differential and Difference Equat.* Folia FSN Univ. Masaryk. Brunensis, Mathematica 13, 2003, 77–86.

[8] CHVALINA J., CHVALINOVÁ, L. *Transposition hypergroups formed by transformation operators on rings of differentiable functions*. Italian Journ. of Pure and Apl. Math., N. 15–2004, p. 93–106.