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Inner Law and Models for Forecasting the Results of Air Defense Battle

V. Gorodnov[1*] and V. Ovcharenko[2]

[1] *National Academy of the National Guard of Ukraine, Kharkiv, Ukraine*
[2] *Institute of the National Guard of Ukraine, Kyiv, Ukraine*

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Abstract:

The purpose of the article was to find the inner properties and regularities of anti-aircraft missile grouping's air defense battle, and based on these regularities to develop and evaluate the adequacy of models as tools for forecasting the results of a battle. To achieve the goal, an example of a real battle and a special model development method were used. The method made it possible to identify the main properties and the inner law of battle, to select a statistically valid mathematical modeling tool in the class of Markov processes with continuous time and discrete states, to develop the desired models with an analytical representation of the inner law of battle, to check their adequacy and to evaluate the accuracy of forecasting by models using the example of a real air defense battle.

Keywords:

air defense, grouping of surface-to-air missile forces, Markov processes, modeling

1 Introduction

In the course of hostilities, the task of protecting important objects of the state and groupings of troops from air strikes is assigned to the anti-aircraft missile (AAM) forces as part of anti-aircraft missile brigades, regiments and subunits, which are deployed in advance in battle formations in positional areas and form an anti-aircraft missile grouping (AAMG). The armament of each subunit includes a surface-to-air missile system (SAMS) of a specific type with a stock of anti-aircraft guided missiles, means of radar reconnaissance and identification of aircraft nationality, means of receiving target designation and communication means.

The parameters of the combat order of the AAM grouping are determined by the requirement to form an entire high-altitude zone of fire in the defended airspace, with the provision of mutual cover for subunits at extremely low altitudes and with a sufficient

* *Corresponding author: National Academy of the National Guard of Ukraine, 3 Zakhysnykiv Ukrainy Square, UA-61001, Kharkiv, Ukraine. Phone: +380 67 954 92 16, E-mail: vgor46@ukr.net. ORCID 0000-0001-8593-8871.*

Nomenclature

Bounds of intervals in the sample $\Delta t_{MAA.i}$	a_{Δ}	Probability of the battle model state S_{ij}	P_{ij}
Bounds of intervals in the sample $\Delta t_{SAM.i}$	b_{Δ}	First derivative of the state probability P_{ij}	$\dot{P}_{ij}$
Increment of normalized value parameter	$D(n^*)$	Probability of aircraft shot down	P_{sda}
Second remarkable limit (e = 2,71828…)	e	Number of freedom degrees	r
Intensity of the enemy aircraft flow	I	Model state when i SAM systems are damaged and j SAMS are in the battle	S_{ij}
Intensity of MAA flow for one SAMS	I_1	Current time	t
Applying the math expectation operation to a random variable in square brackets	$M\,[N_{asd}]$	Mathematical expectation of the separate fire contact duration	T_{avr}
Mathematical expectation operation of the damaged SAM systems number	$M\,[n_{d.S}]$	Interval end time	$t_{end.i}$
Conditional math expectation of j number events in the presence of event i	$M[j\|i]$	Duration of separate fire contact	T_{random}
Normalized value of the damaged SAM systems number math expectation	n^*	Interval start time	$t_{st.i}$
Derivative of math expectation normalized value of damaged SAMSs' number	$\dot{n}^*$	Airstrike duration	T_{str}
Derivative of math expectation normalized value of fire contacts' number	$\dot{n}^*_{fc}$	Normalized value of math expectations of fire contacts and parties' losses number	w
Mathematical expectation of the battle contacts' normalized number	n^*_{fc}	Significance level in assessing the truth of the result	α
Normalized value of the number of the mathematical expectation of shot down aircraft	N^*_{sda}	Relative error value	Δ
Initial number of SAM systems in air defense grouping	n_0	The time interval of the i-th MAA appearance	$\Delta t_{MAA.i}$
Mathematical expectation of the damaged SAM systems number	$n_{d.S}$	The duration of shooting on i-th MAA by SAM system	$\Delta t_{SAM.i}$
Mathematical expectation of the damaged SAM systems number during one fire contact	$n_{d.S.1}$	Parameter of the exponential distribution law	μ
Mathematical expectation of separate battle contacts number	n_{fc}	System load factor	ρ
Mathematical expectation of the fire contacts limiting number until the moment of damaged all SAM systems in grouping	$n_{fc\,\infty}$	Pearson's criterion	χ^2
The amount of MAA in airstrike	N_{MAA}	Critical distribution point of χ^2	$\chi^2_{Critical}$
Mathematical expectation of shot down aircraft number	N_{sda}	A measure of the degree coincidence of exponential distribution law and the distribution law in the sample of time intervals between MAA in an airstrike	$\chi^2_{Exp.MAA}$
Mathematical expectation of downed MAA's limiting number until the moment of all SAM systems in grouping are damaged.	$N_{sda\,\infty}$	A measure of the degree coincidence of exponential distribution law and the distribution law in the sample of fire contacts duration	$\chi^2_{Exp.SAMS}$
Mathematical expectation of shot down aircraft number during one fire contact	$N_{sda.1}$	A measure of the degree coincidence of normal distribution law and the distribution law in the sample of time intervals between MAA in an airstrike	$\chi^2_{Norm.MAA}$
Conditional probability of j-th event when i-th event occurs	$P(j\|i)$	A measure of the degree coincidence of normal distribution law and the distribution law in the sample of fire contacts duration	$\chi^2_{Norm.SAMS}$
Probability of damage to SAM system during battle contact	P^*	the lesion intensity in the presence of q defeated SAMs	η_q

coefficient of the AAM systems zones of fire overlapping in the conditions of their possible destruction.

In turn, the air adversary, as a rule, has some information about the combat formation of the AAM grouping, and plans the parameters of the air strike – the composition, combat formation of means of air attack (MAA) and the procedure for overcoming the defended airspace, using radar counteraction and SAM systems fire damage with the entry into the SAM subunits zones of fire and without it.

The AAM grouping performs a combat mission by conducting air defense battle, which includes a set of consecutive and simultaneous air defense battles (fire contacts) of the SAM subunits with air targets. The moments of the beginning and end of each fire contact and the possible losses of parties are not known in advance, which makes each realization of fire contact and air defense battle of AAM grouping as a whole unique.

When building the battle formation of an AAM grouping, the practical problem of forecasting the values of the following parameters is solved: the expected losses of SAM systems during an air defense battle, assessing the number of fire contacts, the sufficiency of the SAM subunits in battle formation to repel the first and subsequent attacks of enemy's MAA, taking into account possible losses of SAM systems, assessing the sufficiency of the stock of anti-aircraft guided missiles, as well as assessing the losses and capabilities of an air enemy to deliver repeated air strikes. The noted parameters are used to evaluate the performance indicators of the AAM grouping.

As a result, the actual problem arises of finding a stable inner regularity of air defense battle and, on its basis, building models that are adequate enough to forecast the values of noted indicators.

2 Preliminaries and Related Works

According to the composition of the mathematical tools used, the well-known works on modeling the actions of SAM systems for air defense and anti-missile defense can be divided into several categories. Thus, when constructing models for assessing the effectiveness of air defense and missile defense of ground objects [1-4] and a grouping of surface ships [5], the mathematical apparatus of queuing theory [1, 5], game theory [2], Petri nets [4], as well as the idea of heterogeneous networks [3] were used.

At the same time, the authors intuitively believe that the chosen mathematical apparatus corresponds to the processes of air defense battle of AAM grouping. However, the listed mathematical methods do not take into account the main factor of air defense battle – the possibility of SAM systems damage, which makes the noted models devoid of adequacy to real combat processes.

Therefore, the purpose of the article is to identify the essential inner properties and regularity of air defense battle of AAM grouping, and based on these results, to develop and assess the adequacy of the relevant models as tools for forecasting the results of the battle.

To achieve the noted goal, we have used elements of a special technology for models' development [6] and introduced a system of notations (Nomenclature) for variables, which we will use in further reasoning.

3 Research Results

3.1 The Main Features of the Air Defense Force SAM Grouping Battle

In order to identify the main features of the real process, let us consider the dynamics and results of air defense battle in the Suez Canal zone on June 30, 1970 [7], presented

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in Fig. 1 and in Tab. 1. In order to gain air superiority in this region, at 18.31, an air enemy suddenly imposed a battle on an AAM grouping consisting of thirteen SA-2 type SAM systems and three SA-3 type SAM systems, reinforced with portable SAM systems subunits and deployed in battle formation (Fig. 1a).

Let's briefly trace the battle development. The first group of MAA, consisting of 4 aircraft (2 – Skyhawk and 2 – Phantom), at an altitude of 50 meters, under the cover of the terrain, entered the grouping's deployment area. One aircraft was shot down by a portable SAM system, the second aircraft was unsuccessfully shot upon by SAMS #12 SA-3 type. The non-hit aircraft of the first group attacked and hit the SA-2 SAMS #13 and left the zone of fire of AAM grouping.

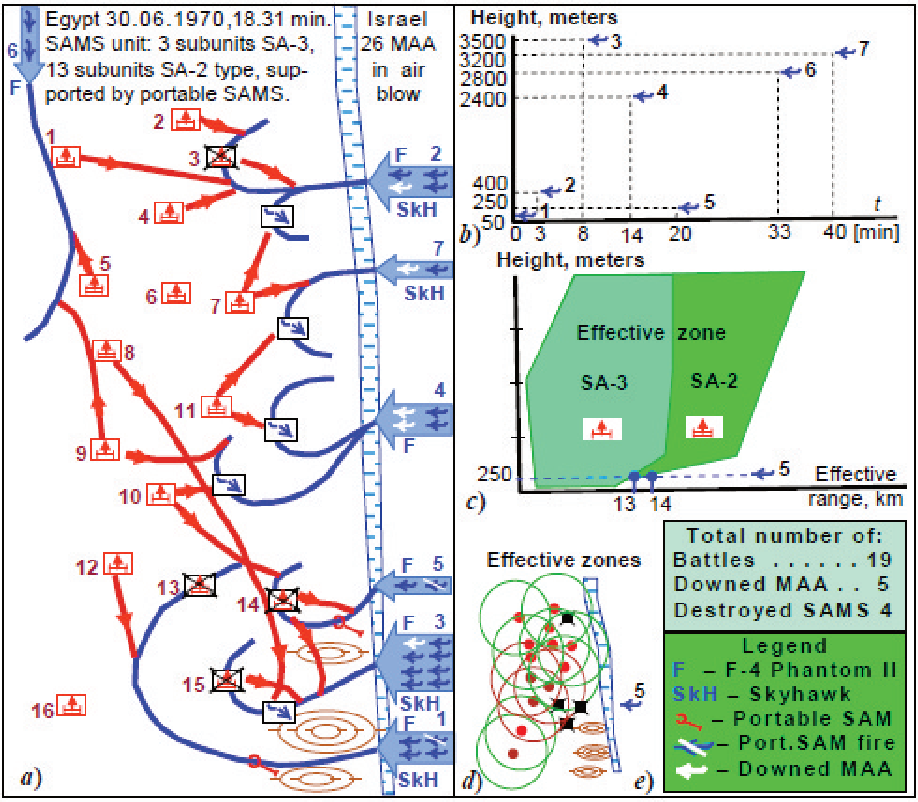

Fig. 1 Scheme and parameters of a real air defense battle on 30.06.1970 *in the Suez Canal zone: a) a map-diagram of the battle dynamics; b) the MAA strike's height-time diagram; c) the SAM affective areas structure; d) total affective area of SAM unit at MAA No 5 flight altitude; e) final battle results and designations on the map-diagram* [7]

At this time, the subunits of the AAM grouping were put on alert. Therefore, the second group of aircraft, which entered the fire zone of grouping on the opposite flank and at an altitude of 400 meters, was met by fire from two SA-2 SAMS (#3 and #7). One of the attacking planes was shot down by SA-2 SAMS #7. Its partner refused to carry out the combat mission, turned around and quickly left the grouping's zone of fire. The remaining two aircraft were unsuccessfully shot upon by SA-2 SAMS #4 and

SA-2 SAMS #1, continued flying, attacked and hit SA-2 SAMS #3, then they were unsuccessfully shot upon by SA-2 SAMS #2 and quickly left the groupings fire zone.

The further development of the battle can be traced according to the map-scheme (Fig. 1a), according to the height-time diagram (Fig. 1b) and according to Tab. 1. Each time, depending on the altitude of aircraft flight, the fire zone range of each SAM systems was changed in accordance with its characteristics (Fig. 1c), which led to changes in the grouping's effective zones coefficient of overlapping (Fig. 1d) and to changes in the degree of aircraft accessibility to be shot upon by AAM grouping.

Tab. 1 Estimates of a real (Fig. 1) *battle's current and integral parameters*

#	The SAMS unit air defense battle events time, minutes			Battle participants' numbers		Losses in the battle contact	#	Integral parameters of SAMS unit's air defense battle	
	Start (&MAA)	End	Δt_{SAMS}	SAMS	MAA				
1	2.00	3.00	4.00	5	6	7	8	9	10
1	0.50	1.23	0.73	Portable	1	1 aircraft	25	n_{fc}	22
2	2.02	2.93	0.91	12	1	—	26	$n_{d\,SAMS}$	4
3	3.28	3.63	0.35	13	1	SAMS #13	27	N_{sda}	5
4	3.02	3.70	0.68	3	2	—	28	P^*	0.181818
5	2.96	4.03	1.07	7	2	1 aircraft	29	P_{sda}	0.227273
6	3.47	4.13	0.66	4	2	—	30	n_0	16
7	2.40	4.30	1.90	1	2	—	31	$D(n_{fc})$	0.011364
8	3.81	4.43	0.62	3	2	SAMS #3	32	$D(n_{d\,SAMS})$	0.062500
9	4.53	4.93	0.40	2	2	—	33	$D(N_{sda})$	0.058000
10	7.74	8.63	0.89	14	3	—	34	$n_{fc\,\infty}$	88
11	8.42	8.73	0.31	15	3	—	35	$N_{sda\,\infty}$	20
12	6.84	9.10	2.26	8	3	1 aircraft	Estimates with inner law		
13	8.82	9.43	0.61	15	3	SAMS #15	36	w (44)	0.251485
14	14.26	15.11	0.85	11	4	1 aircraft	37	n_{fc}	22.130652
15	15.23	15.80	0.57	10	4	1 aircraft	38	$n_{d.S}$	4.023755
16	14.70	16.23	1.53	9	4	—	39	N_{sda}	5.029694
17	20.08	20.40	0.32	14	5	—	40	Δ	0.593873%
18	20.60	20.60	0.25	Portable	5	1 aircraft	41	$\chi^2_{Exp.SAMS}$	3.945236
19	20.63	21.00	0.37	14	5	SAMS #14	42	$\chi^2_{Norm.SAMS}$	7.597282
20	19.66	21.53	1.87	10	5	—	43*	$\chi^2_{Exp.MAA}$	1.675774
21	34.08	34.47	0.39	5	6	—	44*	$\chi^2_{Norm.MAA}$	32.963587
22	33.46	35.10	1.64	9	6	—	45*	α	0.050000
23	40.25	40.50	0.25	7	7	—	46*	$\chi^2_{Critical}$ $(r = 1, \alpha)$	3.890000
24	39.92	41.10	1.18	11	7	1 aircraft	47*	$\chi^2_{Critical}$ $(r = 2, \alpha)$	5.990000

*Note. $\chi^2_{Norm.SAMS} = 7.597 > 3.89$ $\chi^2_{Norm.MAA} = 32.96 > 3.89$ $a_\Delta = (0.02; 0.263; 0.4; 1.08; 6.83)$
$\chi^2_{Exp.SAMS} = 3.945 < 5.99$ $\chi^2_{Exp.MAA} = 1.675 < 5.99$ $b_\Delta = (0.25; 0.53; 0.84; 1.34; 2.26)$

With a short break, the fight lasted 44 minutes. As a result (Fig. 1g), subunits of AAM grouping made 22 fire contacts (Tab. 1 No 25). Out of the 26-enemy aircraft, 5 aircraft were shot down (Tab. 1 No 27) by SA-2 and SA-3 type SAMS subunits and two aircraft were shot down by portable SAMS. Air enemy managed to hit 4 SAM systems (Tab. 1 No 26).

The considered description (Fig. 1 and Tab. 1) makes it possible to select the following significant features of the AAM grouping air defense battle.

From the point of AAM grouping view, MAA operated in groups with a previously unknown (random) strength, in a previously unknown place and time of entry into the fire zone of AAM grouping and individual SAM systems.

The range of each SAM system's fire zone was determined by its characteristics (Fig. 1c) and changed each time, depending on the altitude of enemy aircraft, which led to unpredictable changes in the grouping's total fire zone (Fig. 1d) and its overlap coefficient.

Enemy aircraft, at best, carried out one attack upon SAMS, and did not remain in the zone of fire for repeated battles, which allows us to assert the presence of fuel and ammunition supply on the MAA board based on one attack on a ground target, since an increase in weight reduces MAA maneuverability and increases its vulnerability.

The main repetitive element of the AAM grouping's battle is the air defense battle (fire contact) of a single SAMS subunit. The start and the end times and the results of each battle are not known in advance – they are random. At the same time, the outcomes of each battle can be:

- an enemy aircraft downed (with a probability of P_{sda}),
- an enemy aircraft not downed (with a probability of $1 - P_{sda}$),
- a SAM system destroyed (with probability of P^*),
- a SAM system not destroyed (with probability of $1 - P^*$),
- combinations of outcomes 1-4.

The main parameters that determine the development of AAM grouping air defense battle in time are random time intervals between the start of individual fire contacts (SAMS subunits air defense battles) and the duration of such contacts.

3.2 Inner Law of Air Defense Battle

The considered example allows us to formulate a system of hypotheses about the AAM grouping's air defense battles' essential properties.

- The result of each battle of a SAMS subunit is not known in advance (random) and may include both the defeat of an enemy aircraft and the defeat of SAM system.
- Enemy aircraft have a fuel and ammunition supply on board per one attack, and do not accumulate in the AAM grouping fire zone.
- Each battle of AAM grouping subunit develops in time as a random process, for which the start and the end points are not known in advance (are random).
- In the common zone of fire of several SAM systems, the shot upon the next enemy aircraft is possible by any free SAM system (the effect of mutual assistance in the AAM grouping).
- The combat order of enemy aviation may include groups of aircraft operating sequentially and simultaneously, creating a random flow of "requests for service" for the AAM grouping, with an unknown number of aircraft in each group (the property of MAA's not ordinary flow).
- In the general zone of fire, an aircraft can be shot upon only by that free SAM system, through the zone of fire of which the aircraft trajectory passes, which leads to the effect of incomplete accessibility of the air defense grouping.

To search for the inner law of AAM grouping air defense battle, we will examine the first three hypotheses about its most essential properties. To this end, we will find the mathematical expectation of the enemy aircraft shot down number $N_{sda.1}$ and the SAMS damaged number $n_{d.S.1}$, first for one air defense battle of the SAMS subunit, as the limit to which the mean value of a sample of random variables tends when the sample size tends to infinity, and whose useful properties are known, given, for example, in [8]

$$N_{\text{sda.1}} = 1 \cdot P_{\text{sda}} + 0 \cdot (1 - P_{\text{sda}}) = P_{\text{sda}} \qquad n_{\text{d.S.1}} = 1 \cdot P^* + 0 \cdot \left(1 - P^*\right) = P^* \tag{1}$$

The desired variables coincided with the probability of shooting down an enemy aircraft and hitting SAM system, respectively.

Then let us suppose that by the time t of the battle, an average number of SAMS' battles equaled to n_{fc}. For convenience, let us use complete and simplified notations of the mathematical expectation of the enemy planes number, downed by time t, as $M[N_{\text{sda}}] = N_{\text{sda}}$ and damaged SAM systems as $M[n_{\text{d.S}}] = n_{\text{d.S}}$, and we obtain

$$M[N_{\text{sda}}] = M\left[\sum_{i=1}^{n_{\text{fc}}} N_{\text{sda.1}}\right] = \sum_{i=1}^{n_{\text{fc}}} M[N_{\text{sda.1}}] = \sum_{i=1}^{n_{\text{fc}}} P_{\text{sda}} = P_{\text{sda}} n_{\text{fc}} \text{ then } N_{\text{sda}} = P_{\text{sda}} n_{\text{fc}} \tag{2}$$

$$M[n_{\text{d.S}}] = M\left[\sum_{i=1}^{n_{\text{fc}}} n_{\text{d.S.1}}\right] = \sum_{i=1}^{n_{\text{fc}}} M[n_{\text{d.S.1}}] = \sum_{i=1}^{n_{\text{fc}}} P^* = P^* n_{\text{fc}} \text{ then } n_{\text{d.S}} = P^* n_{\text{fc}} \tag{3}$$

Next, let us take into account the limited number n_0 of SAM systems within the grouping of SAM unit and the condition when every SAM system may be damaged by fire of a hostile aircraft during each separate air-defense battle. And let us tend the fight time to infinity, provided that SAM subunits have an unlimited anti-aircraft guided missiles number and the number of hostile planes, that take turns entering the battle, is not limited (Fig. 2) either.

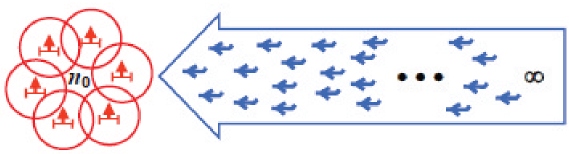

Fig. 2 Air Battle Limit Conditions

Then, with time, all n_0 SAM subunits within the SAM grouping will be damaged. At that moment, the number of air-defense battles n_{fc} will reach its limit $n_{\text{fc}\,\infty}$ value

$$\lim_{t \to \infty} n_{\text{d.S}}(t) = n_0 \qquad \lim_{t \to \infty} n_{\text{fc}}(t) = n_{\text{fc}\infty} \tag{4}$$

Let us substitute the limiting values of $n_{\text{d.S}}$ and n_{fc} from Eq. (4) into the left and right part of expression Eq. (3) and find an estimate of the mathematical expectation of the maximum number of separate air defense battles until the moment of defeat of all the SAM systems within the SAM subunits grouping, as well as the maximum number of air defense battles for one SAM subunit.

$$n_0 = P^* n_{\text{fc}\infty} \quad \text{then} \quad n_{\text{fc}\,\infty} = \frac{n_0}{P^*} \quad \text{and} \quad n_{\text{fc}\,\infty}(n_0 = 1) = \frac{1}{P^*} \tag{5}$$

It is not difficult to verify the correctness of the Eq. (5) physical meaning. If the probability of a SAM subunit defeat in one air defense battle is equal to one, i.e., $P^* = 1$, then the mathematical expectation of separate air defense battles number will coincide with the number of SAM subunits n_0 within the SAM subunits grouping.

Substituting the value of the variable $n_{\text{fc}\infty}$ from Eq. (5) into the right part of Eq. (2), let us find the limiting value of the mathematical expectation of number $N_{\text{sda}\,\infty}$

of downed enemy planes until the moment of all SAM subunits damaged within the SAM subunits grouping:

$$N_{\text{sda}\infty} = P_{\text{sda}} n_{\text{fc}\infty} = n_0 \frac{P_{\text{sda}}}{P^*} \tag{6}$$

Let us use the main ideas of [6] and turn to the relative variables of the separate air defense battles number and the casualties of the parties:

$$n_{\text{fc}}^*(t) = \frac{n_{\text{fc}}(t)}{n_{\text{fc}\infty}} \quad N_{\text{sda}}^*(t) = \frac{N_{\text{sda}}(t)}{N_{\text{sda}\infty}} \quad n^*(t) = \frac{n_{\text{d.S}}(t)}{n_0} \tag{7}$$

If we divide the left and right sides of the right expression in Eq. (3) by the number n_0 of SAM subunits within the grouping's SAM subunits and take into account the Eq. (7), we can obtain an unexpected result – the Eq. (8) of the relative values of the mathematical expectations of damaged SAM subunits' number $n^*(t)$ and the number of separate air defense battles $n^*_{\text{fc}}(t)$

$$\frac{n_{\text{d.S}}(t)}{n_0} = \frac{P^* n_{\text{fc}}(t)}{n_0} = \frac{n_{\text{fc}}(t)}{n_0 / P^*} \quad \text{then} \quad n^*(t) = \frac{n_{\text{fc}}(t)}{n_{\text{fc}\infty}} = n_{\text{fc}}^*(t) \tag{8}$$

Let us multiply the numerator and denominator of the right side in the last Eq. (8) by the probability P_{sda} of the enemy plane shot down as a result of a separate air defense battle, and find Eq. (9):

$$n^*(t) = \frac{P_{\text{sda}} n_{\text{fc}}(t)}{P_{\text{sda}} n_{\text{fc}\infty}} = \frac{N_{\text{sda}}(t)}{N_{\text{sda}\infty}} = N_{\text{sda}}^*(t) \tag{9}$$

Equations (8) and (9) are obtained on the basis of the most essential hypotheses 1-3 for the battle processes by using admissible operations with mathematical expectations [8] of random variables which allow us to formulate an inner law of air defense battle.

A consequence of hypotheses 1-3, which concern the most significant features of air defense battles, is the equality of the relative values of the mathematical expectations of the number of separate air defense battles and the casualties of the parties at any moment of air defense battle.

$$n^*(t) = N_{\text{sda}}^*(t) = n_{\text{fc}}^*(t) = w(t) \tag{10}$$

In this case, the absolute values of these parameters can have different meanings depending on the conditions of the battle.

For the extreme values of time t, it is easy to verify Eq. (10). Indeed, at the beginning of the battle, all variables (7) in Eq. (10) are strictly equal to zero. At the time of all SAM subunits' defeat, the values of all these variables become equal to one.

If in the process of verification of the model developed below, Eq. (10) of the relative casualties of the parties is/were found, such a model can be considered adequate to a real air defense battle with the accuracy of hypotheses 1-3 on the most essential properties of air defense battle processes. Otherwise, the model adequacy becomes doubtful and the application of such a model becomes inappropriate.

Let's go back to the battle example (Fig. 1, Tab. 1) and note that at the moment of each end of the battle, the normalized values in Eq. (7), in the event of an enemy aircraft and/or SAM system being hit, will increase (Fig. 3) by the corresponding value (Tab. 1, No 31-33):

$$D\left(n_{\text{fc}}^{*}\right)=\frac{1}{n_{\text{fc}\,\infty}}=\frac{P^{*}}{n_0} \qquad D\left(n^{*}\right)=\frac{1}{n_0} \qquad D\left(N_{\text{sda}}^{*}\right)=\frac{1}{N_{\text{sda}\infty}}=\frac{P^{*}}{n_0 P_{\text{sda}}} \tag{11}$$

We use the least squares method and obtain smoothed representations of these values and the relative losses $w(t)$ of the sides (Fig. 3) for the considered (Fig. 1) example

$$\left.\begin{aligned} n_{\text{fc}}^{*}(t)&=0.0831\cdot\ln t-0.0703 \qquad & N_{\text{sda}}^{*}(t)&=0.0886\cdot\ln t-0.0976 \\ n^{*}(t)&=0.0880\cdot\ln t-0.0604 \qquad & w(t)&=0.0866\cdot\ln t-0.0761 \end{aligned}\right\} \tag{12}$$

Equations (5)-(7) make it possible to find estimates of the absolute values for the mathematical expectations of the number of air defense battles and losses of the parties

$$n_{\text{fc}}(t)=w(t)n_{\text{fc}\infty} \qquad N_{\text{sda}}(t)=w(t)N_{\text{sda}\infty} \qquad n_{\text{d.S}}(t)=w(t)n_0 \tag{13}$$

At the time of the end of air attack repelling (t = 44 minutes), you can find the value of relative $w(44)$ and absolute losses of the parties (Tab. 1, No 36-39), which turn out to be overestimated by 0.59 % (Tab. 1, No 40) relative to real values due to the lack of the fifth and sixth hypotheses consideration.

However, the fact of a relatively exact coincidence of the estimates obtained with the real results, testifies in favor of the correctness of the found air defense battle inner law Eq. (10). In order to obtain the possibility of a practical application of the found law, it is necessary to find a variant of its analytical description.

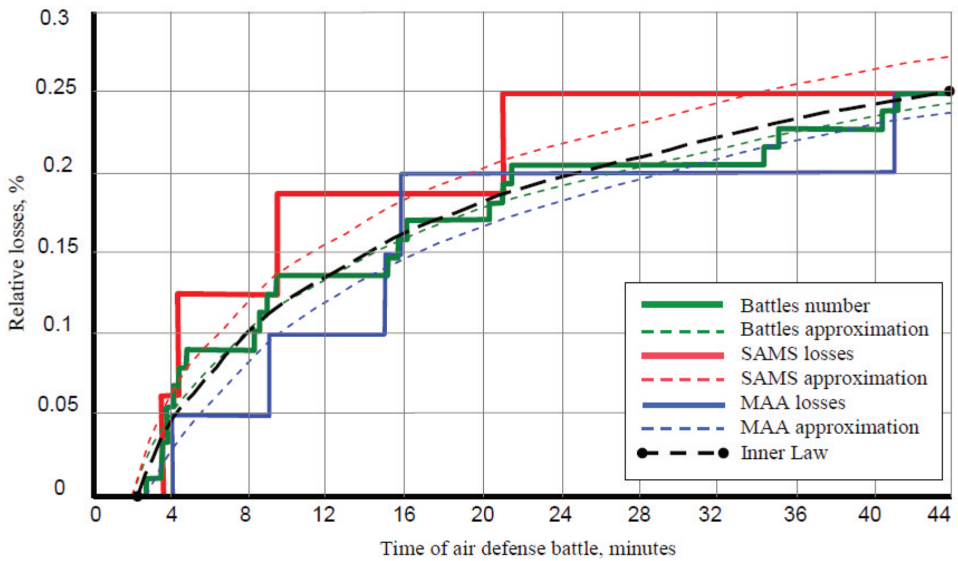

Fig. 3 Dependence of the sides' relative losses on the battle time

3.3. The Simplest Model of Separate Air Defense Battles

The adequacy of the battle model is possible [6] only when choosing a mathematical apparatus that corresponds to the laws of distribution of its main random parameters that determine the development of the battle – the intervals between the entry of the MAA into the fire zone (Tab. 1, column 2, $\Delta t_{MAA.i} = t_{st.i+1} - t_{st.i}$) and the duration of the SAM systems firing cycle (Tab. 1, column 4; $\Delta t_{SAM.i} = t_{end.i} - t_{st.i}$).

The results of evaluating possible laws of distribution for each sample of marked random variables according to the Pearson criterion (Tab. 1, No 45-47 and Note – criterion χ^2) determine the need to reject the hypothesis of their normal distribution and to accept the hypothesis of their exponential integral distribution law.

As a result, the model of combat operations (sequences of battles) of a separate SAMS subunit should be built in the class of Markov processes with continuous time and discrete states.

In order to find an analytical description of the parties normalized losses parameter $w(t)$, as the first step, we will build a model for the key element to fight off the enemy MAA blow – for an air defense battles sequence of one-channel SAM system with enemy aircraft flow of intensity I single planes per minute.

In order to build a model, let us list possible states S_{ij} of SAM subunit during the battle (during fight off the enemy air blow), essential for the purposes of its actions, that is, those states that differ in the possibility of opening fire at the "next" MAA. In the state designation S_{ij}, the first index i is used to indicate the number of damaged SAM subunits in this state, the second index j is used to indicate the number of hostile airplanes being fired at in this state:

- S_{00} – SAM subunit is not damaged, and is free,
- S_{01} – SAM subunit is not damaged, and is firing at one hostile airplane,
- S_{10} – SAM subunit is damaged and can't shoot the hostile planes.

We can obtain the diagram of the individual SAM subunit's air defense battle simplest model (Fig. 4), where the transition from the state S_{00} to the state S_{01} is possible upon detection of the next hostile airplane and is characterized by the intensity ("frequency") of the fire contacts I.

Each air defense battle can continue for random time T_{random}, that has the exponential law with the mathematical expectation T_{avr}, with μ parameter and with the intensity I of air defense battles occurrence:

$$M\left[T_{random}\right] = T_{avr} \qquad \mu = \frac{1}{T_{avr}} \qquad I = \frac{N_{MAA}}{T_{str}} \tag{14}$$

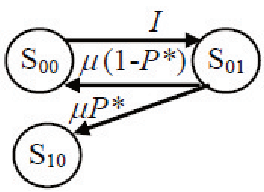

Fig. 4 Diagram of the simplest model of SAM system air defense battle

Each air defense battle can result in the defeat of the SAM system with the probability P^* and the process transition (Fig. 4) from the state S_{01} to the state S_{10}, or with the probability $(1 - P^*)$ it can have a successful outcome for the SAM system, which leads to the transition from the state S_{01} to the state S_{00}.

The exponential distribution of random variables in the battle process under consideration makes it possible to set up a system of Kolmogorov-Chapman differential equations [9] for the probabilities P_{ij} of the states S_{ij} of the battle model (Fig. 4), where, for convenience, let us denote derivatives by a point over the probabilities of the states and omit the dependence of probabilities on time as follows:

$$\dot{P}_{00} = -I \cdot P_{00} + \mu \cdot (1-P^*) \cdot P_{01} \qquad \dot{P}_{01} = -\mu \cdot P_{01} + I \cdot P_{00} \qquad \dot{P}_{10} = \mu \cdot P^* \cdot P_{01} \quad (15)$$

Let us integrate the equations' system (15) under initial conditions:

$$P_{00}(t=0) = 1 \qquad P_{01}(t=0) = P_{10}(t=0) = 0 \quad (16)$$

We will get

$$\left.\begin{aligned} P_{00} &= C_1 e^{\lambda_1 \mu t} + C_2 e^{\lambda_2 \mu t} \\ P_{01} &= \frac{\rho}{\alpha}\left(e^{\lambda_1 \mu t} - e^{\lambda_2 \mu t}\right) \\ P_{10} &= 1 - \frac{1}{\alpha}\left(\lambda_1 e^{\lambda_2 \mu t} - \lambda_2 e^{\lambda_1 \mu t}\right) \end{aligned}\right\} \quad (17)$$

where

$$\rho = \frac{I}{\mu} \quad \alpha = \sqrt{1+\rho^2+2\rho\left(1-2P^*\right)} \quad C_{12} = \frac{\alpha \pm (1-\rho)}{2\alpha} \quad \lambda_{12} = \frac{\pm\alpha - (1+\rho)}{2} \quad (18)$$

The mathematical expectation of the separate air defense battles number n_{fc} (of the attacked enemy airplanes) by the time of the fight off enemy air strike t will be determined taking into account the time of the SAM system's stay in the occupied state S_{01} and its "productivity" μ

$$n_{fc}(t) = \mu \int_0^t P_{01}(t)\,d\tau = \frac{1}{P^*}\left[1 - \frac{1}{\alpha}\left(\lambda_1 e^{\lambda_2 \mu t} - \lambda_2 e^{\lambda_1 \mu t}\right)\right] \quad (19)$$

3.4. Verification of the Air Defense Battle Simplest Model

To verify the simplest model Eqs (17)-(19), let's find the mathematical expectation of air defense battles' maximum possible number for the entire time up to the moment of the SAM system's defeat. It can be found, passing to the limit in the Eq. (19)

$$n_{fc\infty} = \lim_{t\to\infty} n_{fc}(t) = \frac{1}{P^*}\left(1 - \frac{1}{\alpha}\cdot 0\right) = \frac{1}{P^*} \quad (20)$$

Let us note that the resulted Eq. (20) has coincided with the value of Eq. (5) obtained earlier for this variable concerning one SAM system.

In order to verify the adequacy of the battle simplest model Eqs (17)-(19) of an individual SAM system, let us find the relative value of the separate air defense battles mathematical expectation number at any time t of the battle. To this end, let us divide the left side of Eq. (19) into the left side of Eq. (20) and the right side of Eq. (19) by the right side of Eq. (20), then we get

$$\frac{n_{fc}(t)}{n_{fc\,\infty}} = \frac{1}{P^*}\left[1 - \frac{1}{\alpha}\left(\lambda_1 e^{\lambda_2 \mu t} - \lambda_2 e^{\lambda_1 \mu t}\right)\right] \cdot \left(\frac{1}{P^*}\right)^{-1} \quad (21)$$

After reducing the same variables, we will find:

$$n_{\text{fc}}^{*}(t) = 1 - \frac{1}{\alpha}\left(\lambda_1 e^{\lambda_2 \mu t} - \lambda_2 e^{\lambda_1 \mu t}\right) \tag{22}$$

The relative value of the mathematical expectation $n^*(t)$ of the damaged SAM systems number at any time of the battle t is found taking into account the known properties of the mathematical expectation of random variables [8]

$$n^*(t) = \frac{n_{\text{d.S}}(t)}{n_0} = \frac{n_{\text{d.S}}(t)}{1} = n_{\text{d.S}}(t) = 0 \cdot \left[P_{00}(t) + P_{01}(t)\right] + 1 \cdot P_{10}(t) = P_{10}(t) \tag{23}$$

Thus, the relative value of the damaged SAM systems mathematical expectation number at any time is equal to the probability of the state P_{10}

$$n^*(t) = P_{10}(t) = 1 - \frac{1}{\alpha}\left(\lambda_1 e^{\lambda_2 \mu t} - \lambda_2 e^{\lambda_1 \mu t}\right) \tag{24}$$

Comparing Eqs (22) and (24), we can see that they are identical:

$$n^*(t) = n_{\text{fc}}^*(t) \tag{25}$$

Using Eqs (2), (6) and (7) we can see that the expression for the relative value of the mathematical expectation $N_{\text{sda}}^*(t)$ of the downed enemy planes number differs from the expression for $n_{\text{fc}}^*(t)$ by the probability of the enemy plane shot down and, at the same time, it coincides with Eq. (19)

$$N_{\text{sda}}^*(t) = \frac{N_{\text{sda}}(t)}{N_{\text{sda}\infty}} = \frac{P_{\text{sda}} n_{\text{fc}}(t)}{P_{\text{sda}} n_{\text{fc}\infty}} = \frac{n_{\text{fc}}(t)}{n_{\text{fc}\infty}} = n_{\text{fc}}^*(t) \tag{26}$$

Based on Eqs (20)-(26), it can be stated that there is the presence of equality of relative values of the mathematical expectations number of fire contacts and losses of the sides

$$n^*(t) = N_{\text{sda}}^*(t) = n_{\text{fc}}^*(t) = P_{10}(t) = w(t) \tag{27}$$

in the analytical description Eqs (15)-(19) of the simplest air defense battle model.

Thus, the obtained simplest model of an individual SAM subunit battle proves to be adequate to a real battle with the accuracy of accepted hypotheses 1-3 concerning the most essential features of air defense battle.

3.5. Model of Fight off the Enemy Air Blow by AAM Grouping

For a rough estimate of the expected results of an air defense battle of AAM force grouping that includes n_0 single-channel on the target SAM systems, and that fights off an enemy blow with an intensity of I planes/minute, the grouping battle model can be replaced by a set of battle models Eqs (15)-(19) of single SAM systems, each of which fights off blows of intensity

$$I_1 = I / n_0 \tag{28}$$

In this case, the overall result can be found using Eqs (27), (24) and (13). However, in this case, the number of shot and destroyed MAA is deliberately underestimated, since it excludes the effect of system performance increasing, which is associated with damping the unevenness of the airplanes input flow in a multi-channel AAM grouping.

Therefore, we have expanded the model shown in Fig. 4. Let us include the AAM grouping n_0 single-channel SAM subunits and temporarily retain the assumption that any enemy aircraft can be shot upon by any SAMS.

In this case, the indices of air defense battle possible states S_{ij} can take on the values $0 \le i \le n_0$, $0 \le j \le n_0$, and the model's graph will take the form shown in Fig. 5a.

To obtain an analytical description suitable for practical calculations, we transform the model (Fig. 5a) into the equivalent model of Fig. 5b.

Let us find the probability P_0 of damaging exactly $i = 0$ SAM systems as the sum of the model's graph zero level probabilities in Fig. 5a, then we find the derivative of this probability $\dot{P}_0$

$$P_0 = \sum_{j=0}^{n_0} j \cdot P_{0\mathrm{j}} \qquad \dot{P}_0 = \sum_{j=0}^{n_0} j \cdot \dot{P}_{0\mathrm{j}} \tag{29}$$

For the product of two events ($i = 0$ and j), we take into account the probability multiplication theorem $P_{0\mathrm{j}} = P_0 \cdot P(j|i = 0)$ and then we substitute it into the right side of the Kolmogorov-Chapman equation for the derivatives of the state probabilities S_{0j} and after transformations we obtain the ability to move to an equivalent model Fig. 5b

$$\begin{aligned} \dot{P}_0 &= -\mu \cdot P^* \cdot P_{01} - 2\mu \cdot P^* \cdot P_{02} - \ldots - n_0 \cdot \mu \cdot P_{0\mathrm{n}_0} = -\mu \cdot P^* \cdot \sum_{j=0}^{n_0} j \cdot P_{0\mathrm{j}} = \\ &= -\mu \cdot P^* \cdot \sum_{j=0}^{n_0} j \cdot P_0 \cdot P(j \,|\, i=0) = -\mu \cdot P^* \cdot P_0 \cdot M[j \,|\, i=0] = \\ &= -\eta_0 \cdot P_0 \end{aligned} \tag{30}$$

where the conditional mathematical expectation of the occupied SAM systems number has the form

$$M[j \,|\, i=0] = \sum_{j=0}^{n_0} j \cdot P(j \,|\, i=0) \tag{31}$$

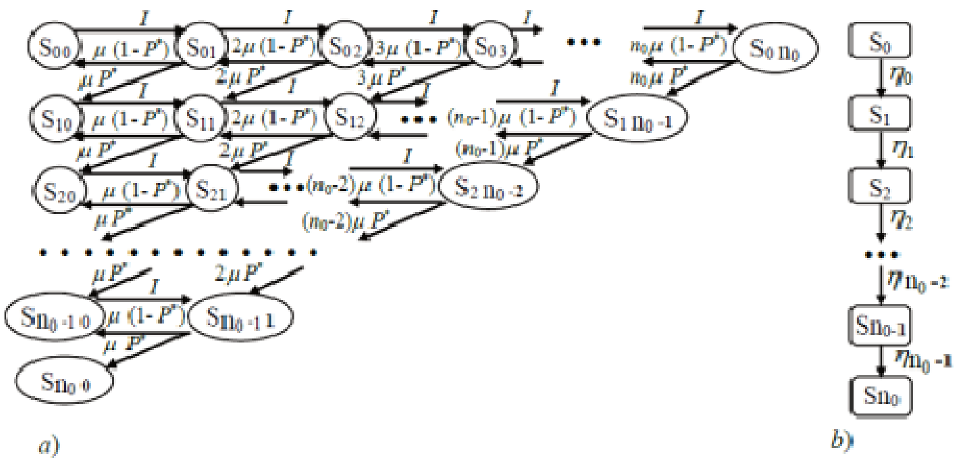

Fig. 5 Graph of the air defense battle's model of a fully accessible grouping of a single-channel by target SAM systems: a) the initial model; b) equivalent model

Arguing similarly, we find a system of differential equations that describe the fully accessible AAM grouping's air defense battle with a flow of enemy single MAA

$$\dot{P}_q = -\eta_q \cdot P_q + \eta_{q-1} \cdot P_{q-1} \qquad q = 1, ..., n_0 - 1 \qquad \dot{P}_{n_0} = \eta_{n_0} \cdot P_{n_0 - 1} \tag{32}$$

where

$$\eta_q = \mu \cdot P^* \cdot M\left[j \mid i = q \right] \tag{33}$$

$$M\left[j \mid i = q \right] = \sum_{j=0}^{n_0 - q} j \cdot P(j \mid i = q) \tag{34}$$

The solution of the equations system Eq. (32) under initial conditions $P_0(0) = 1$, $P_q(0) = 0$, $q = 1, \ldots, n_0$ and constant values of $M[j|i]$ has the form

$$P_0 = e^{-\eta_0 \cdot t} \qquad P_q = \sum_{k=0}^{q} F_{qk} e^{-\eta_k \cdot t} \qquad q = 1, ..., n_0 \tag{35}$$

where functions F_{qk} have the form

$$F_{00} = 1 \qquad F_{qk} = \frac{\eta_{q-1}}{\eta_q - \eta_k} F_{q-1,k} \qquad k = 1, ..., q - 1 \qquad F_{qq} = -\sum_{\tau=1}^{q-1} F_{q\tau} \tag{36}$$

With an insignificant "load" of the AAM grouping, determined by the condition

$$\frac{I}{\mu\left(1 - P^*\right)} \le n_0 - i - 2 \qquad i < n_0 - 2 \tag{37}$$

expression for $M[j|i]$ takes the form

$$M\left[j \mid i \right] = \frac{I}{\mu\left(1 - P^*\right)} \left[1 - e^{-\mu\left(1 - P^*\right) t} \right] \ne f\left(i, j\right) \tag{38}$$

For conditions $t > [I / \mu(1 - P^*)]$ the value $M[j|i]$ is practically independent of the battle time. Then $\eta_q \approx \eta_{q-1} = \eta$, $q = 1, \ldots, n_{0-1}$, and Eq. (35) takes the form

$$P_q = \frac{(\eta t)^q}{T_{\text{avr}}} e^{-\eta t} \qquad q = 0, ..., n_0 - 1 \qquad P_{n_0} = 1 - \sum_{q=0}^{n_0 - 1} P_q \tag{39}$$

3.6. Verification of the Fight off the Enemy Air Blow by AAM Grouping Model

The expressions for the mathematical expectations of the normalized losses for AAM grouping $n^*(t)$ and enemy MAA $N^*(t)$ in the model of fight off enemy MAA blow by AAM grouping (Fig. 5a) will take the form:

$$n^*\left(t\right) = \frac{n\left(t\right)}{n_0} = \frac{1}{n_0} \sum_{i=0}^{n_0} i \sum_{j=0}^{n_0 - i} P_{ij}\left(t\right) \tag{40}$$

$$n_{\text{fc}}^*\left(t\right) = \frac{n_{\text{fc}}\left(t\right)}{n_{\text{fc}\,\infty}} = \frac{P^* \mu}{n_0} \sum_{i=0}^{n_0} \sum_{j=0}^{n_0 - i} j \int_0^t P_{ij}\left(\tau\right) \mathrm{d}\tau \tag{41}$$

After differentiating Eqs (40) and (41), we obtain

$$\dot{n}^{*}(t)=\frac{1}{n_0}\sum_{i=0}^{n_0} i\sum_{j=0}^{n_0-i}\dot{P}_{ij}(t) \tag{42}$$

$$\dot{n}_{\mathrm{fc}}^{*}(t)=\frac{P^{*}\mu}{n_0}\sum_{i=0}^{n_0}\sum_{j=0}^{n_0-i} jP_{ij}(t) \tag{43}$$

The differential Eqs (42) and (43) must be integrated under the same initial conditions

$$n^{*}(0)=0 \qquad\qquad n_{\mathrm{fc}}^{*}(0)=0 \tag{44}$$

The normalized values of individual air defense battles' mathematical expectations number coincide with the normalized values of the downed MAA mathematical expectations number (9). Therefore, to prove the validity of equality (10) in the model (Fig. 5a), it is sufficient to prove the equality of the mathematical expectations of the SAMS losses normalized values and the same number of battles.

Let us assume that if there are n_0 single-channel SAM systems in the AAM grouping, the differential equations for the mathematical expectations of the SAMS losses normalized values and the number of battles are identical

$$\dot{n}^{*}(n_0,t)=\dot{n}_{\mathrm{fc}}^{*}(n_0,t) \tag{45}$$

it means

$$\mu P^{*}\sum_{i=0}^{n_0}\sum_{j=0}^{n_0-i} jP_{ij}(t) = \sum_{i=0}^{n_0} i\sum_{j=0}^{n_0-i}\dot{P}_{ij}(t) \tag{46}$$

and let us prove that in this case [Eqs (45), (46)] the identity of the differential equations for mathematical expectations of the SAMS normalized losses and the number of battles will remain even in the case of increasing the number of single-channel SAMS in AAM grouping by one, i.e. it will be true the equality

$$\dot{n}^{*}(n_0+1,t)=\dot{n}_{\mathrm{fc}}^{*}(n_0+1,t) \tag{47}$$

For the aim of brevity, we omit the notation of the state probabilities dependence on time. To prove the validity of Eq. (47), it suffices to prove the validity of the equivalent equality

$$\mu P^{*}\sum_{i=0}^{n_0+1}\sum_{j=0}^{n_0+1-i} jP_{ij} = \sum_{i=0}^{n_0+1} i\sum_{j=0}^{n_0+1-i}\dot{P}_{ij} \tag{48}$$

So, for $n_0 = 1$ we use the system of Eq. (15) and for (46) we obtain

$$\dot{n}^{*}(t)=\frac{1}{n_0}\sum_{i=0}^{n_0} i\sum_{j=0}^{n_0-i}\dot{P}_{ij}(t)=\dot{P}_{10}(t)=\mu P^{*}P_{01} \tag{49}$$

$$\dot{n}_{\mathrm{fc}}^{*}(t)=\frac{P^{*}\mu}{n_0}\sum_{i=0}^{n_0}\sum_{j=0}^{n_0-i} jP_{\mathrm{ij}}(t)=P^{*}\mu\left[1\cdot P_{01}(t)\right]=\mu P^{*}P_{01} \tag{50}$$

Comparing the right parts of the obtained equations, we are convinced of their identity.

When increasing n_0 to n_0+1 in the graph of model of air defense battle (Fig. 5a), additional states S_{i,n_0+1-i}, $i=0, \ldots, n_0+1$ and additional connections of states with new states S_{i,n_0-i}, $i=0, \ldots, n_0$ appear.

So, for $n_0 = 2$, from the equations for the states of the model's graph Fig. 5a we find

$$\sum_{i=0}^{2} i \sum_{j=0}^{2-i} \dot{P}_{ij}(t) = 1 \cdot \left[\mu P^* P_{01} + 2\mu P^* P_{02} - \mu P^* P_{11} \right] + 2\mu P^* P_{11} = \\ = \mu P^* P_{01} + \mu P^* \sum_{i=0}^{1} (1+1-i) P_{i;1+1-i} \tag{51}$$

$$\mu P^* \sum_{i=0}^{2} \sum_{j=0}^{2-i} j P_{ij} = \mu P^* \left[(1 \cdot P_{01} + 2 \cdot P_{02}) + (1 \cdot P_{11}) \right] = \\ = \mu P^* P_{01} + \mu P^* \sum_{i=0}^{1} (1+1-i) P_{i;1+1-i} \tag{52}$$

Comparing the right parts of the obtained equations, we are convinced of their identity and the possibility of an additive increment in the right part of Eqs (51) and (52).

In general case, with an increase in the number of SAM systems in a grouping by one, the left and right parts of equality (48) take the form of Eqs (53) and (54), respectively, with the same additional members

$$\mu P^* \sum_{i=0}^{n_0+1} \sum_{j=0}^{n_0+1-i} j P_{ij}(t) = \mu P^* \sum_{i=0}^{n_0} \sum_{j=0}^{n_0-i} j P_{ij}(t) + \mu P^* \sum_{i=0}^{n_0} (n_0+1-i) P_{i;n_0+1-i}(t) \tag{53}$$

$$\sum_{i=0}^{n_0+1} i \sum_{j=0}^{n_0+1-i} \dot{P}_{ij}(t) = \sum_{i=0}^{n_0} i \sum_{j=0}^{n_0-i} \dot{P}_{ij}(t) + \mu P^* \sum_{i=0}^{n_0} (n_0+1-i) P_{i;n_0+1-i}(t) \tag{54}$$

The identity of the differential equations for the mathematical expectations of the number of individual air defense battles and the normalized losses of SAM systems was found at $n_0 = 1$, at $n_0 = 2$, and was proved for general case of an increase in the number of SAM systems in a grouping by one unit Eqs (53), (54), which allows us to inductively claim the validity of the equality of mathematical expectations of the parties normalized losses (10) for the model of fight off the MAA air enemy blow by the AAM grouping (Fig. 5a) and for its equivalent representation of Fig. 5b.

Thus, in the model of AAM grouping's combat operations, there is the law of equality for the relative values of the mathematical expectations of the party's losses – equality (10) and at the same time, the main hypotheses (1-4) about the most essential properties of air defense battle are taken into account.

Therefore, there is reason to believe that the model of fight off the enemy MAA air blow by the AAM grouping (Fig. 5a) and (Fig. 5b) is adequate to real air defense battle with the accuracy of hypotheses 1-4 about its most essential features.

Note that for the case of AAM grouping of m channel SAM systems in the model of Fig. 5a, the defeat of one SAM system will be accompanied by a decrease in the number of states at the next lower level of the graph not by one state, but immediately by m states. In this case, the model (Fig. 5b) will not change and the presence of equality (10) in the air defense battle model (Fig. 5a) is preserved.

3.7 Verification of Battle Model Suitability for Practical Calculations

We will check the suitability of the developed models for their intended use according to the information of a real air defense battle (Fig. 1, Tab. 1).

The sequence of calculations for the simplest model of a separate SAM subunit air defense battles [model 1, Eqs (5), (6), (7), (13), (14), (18), (24), (28), Tab. 2 No 1-21] allows us to assert the underestimated calculation results of the AAM grouping air defense battle by indicators of mathematical expectations of battles number, damaged SAM systems and downed enemy aircraft (Tab. 2, No 18-20), respectively, which is the reason for the lack of consideration the mutual assistance effect between SAM systems in AAM grouping during shooting enemy's MAA.

However, the indicator $w(44)$ of the error magnitude in calculating of parties' normalized losses (Tab. 2 No 17) relative to its value in a real battle (Tab. 1 No 36) was less than 3 % (Tab. 2 No 21), which allows to use expression (24) as a variant of the analytical representation of inner law of equality between the relative values of the mathematical expectations of parties losses

$$w(t) \approx n^*(t) = P_{10}(t) = 1 - \frac{1}{\alpha}\left(\lambda_1 e^{\lambda_2 \mu t} - \lambda_2 e^{\lambda_1 \mu t}\right) \tag{55}$$

where the elements of Eq. (55) are described in Eq. (18).

Tab. 2 Data and calculation results using battle models

Model 1 Eqs (5), (6), (7), (13), (14), (18), (24), (28)									Model 2 Eqs (33), (38), (39), (40)		
No	Argument	Value	No	Argument	Value	No	Argument	Value	No	Argument	Value
1	T_{str} [min]	44	8	μ [1/min]	1.022	15	$\lambda_1\mu t$	−0.286	22	I [1/min]	0.5909
2	n_0	16	9	$N_{sda\,\infty}$	20	16	$\lambda_2\mu t$	−46.30	23	$M\,[j\|i]$	0.7066
3	P^*	0.1818	10	I_1 [1/min]	0.0369	**17**	$w \approx P_{10}$	**0.2447**	24	η [1/min]	0.1313
4	P_{sda}	0.2273	11	ρ	0.0361	18	n_{fc} (44)	21.537	25	$e^{-\eta t}$	0.00309
5	$n_{fc\infty}$	88	12	α	1.0233	19	$n_{d.S}$ (44)	3.9159	26	$\Sigma(q{\cdot}P_q)$	5.7721
6	$N_{sda\,\infty}$	20	13	λ_1	−0.0063	20	N_{sda} (44)	4.8949	27	$w \approx [\Sigma(q{\cdot}P_q)]/n_0$	**0.3607**
7	T_{avr} [min]	0.9784	14	λ_2	−1.0297	21	Δ *)	2.68 %	28	Δ **)	−43.45 %

*) – 1 SAMS model **) – n_0 SAMS model

The sequence of calculations according to the fight off the enemy air blow by AAM grouping model [model 2, Eqs (5), (6), (13), (14), (33), (38), (39), (40), Tab. 2 No 22-28] allows us to assert the overestimated results of the AAM grouping air defense battle calculations in terms of the sides normalized losses w (44) (Tab. 2 No 27) relative to its value in a real battle (Tab. 1 No 36) by more than 43 %. The discrepancies are explained by the lack of consideration of the fifth and sixth hypotheses about the essential properties of AAM grouping's air defense battle in model 2.

So, the placement of SAM systems on the ground leads to the effect of incomplete accessibility of the AAM grouping's SAM systems, when the next MAA is accessible for firing not to any SAM system, but only to some of them, through the fire zone of which the MAA trajectory passes.

Under these conditions, the strength of the MAA in the group may exceed the number of SAM systems to which this group of MAA is accessible for firing. The noted effects can be taken into consideration when calculating the conditional mathematical expectation Eq. (38), which will make it possible to more accurately take into account the placement of SAM systems in battle formation, the separation of the MAA in height, as well as the possible composition of the MAA in groups.

4 Results

The use of special modeling technology elements [6] made it possible to identify the most essential properties and find inner low Eq. (10) of a real AAM grouping's air defense battle. At the same time, the forecast error, using the revealed law, can be less than 0.6 % (Tab. 1 No 40), which allows using this law for the purposes of decision-making practice. The use of this technology [6] also made it possible to select a mathematical apparatus that is statistically adequate to the battle processes and with the help of which it became possible to build a model of a single SAM system air defense battle and a model of fight off the MAA air blow by an AAM grouping.

Both models describe the dynamics of the battle and have an explicit analytical description of the found inner law of air defense battle, which makes it possible to assert their adequacy to the real air defense battle with the accuracy of accepted hypotheses (1-3 and 1-4 respectively) about its most essential properties.

As a result, there is reason to believe that the objectives of the research have been achieved.

5 Conclusions

A retrospective calculation of a real battle results (Fig. 1) according to the first model shows a high level of agreement between the calculated and real results of the battle (an error is less than 3 %), which also testifies in favor of its adequacy and the possibility of being applied in practice.

However, the models do not yet contain means of accounting for the deployment of SAM systems in the AAM grouping battle formation on real terrain, do not contain means of accounting for interference conditions, do not provide for taking into account the group strength and separation of the MAA in a blow at flight heights, as well as the capabilities of multi-channel SAM systems on the target.

A similar retrospective assessment for the air defense battle model of the AAM grouping shows a significant (more than 42 %) overestimation of the predicted results due to the lack of consideration for the combat conditions noted above.

However, the model is suitable for predicting the dynamics of air defense battle of a multi-channel SAM systems on the target in AAM grouping due to the fact that it contains the parameter $M[j|i]$, which makes it possible to take into account the influence of the SAM systems placement in the AAM grouping battle formation, the group strength and separation of MAA at flight heights in a blow, which may be the content of the corresponding models.

The analytical expressions of the developed models depend on a set of weapons characteristics and on the parameters of the parties' actions tactics, which allows to use models for preliminary estimates of the AAM grouping combat missions effectiveness indicators and for assessing the degree of influence of the weapons characteristics on the expected results of performing tasks.

In addition, for more complete forecasts of the AAM grouping combat mission's performance, initial data on the air enemy are needed – the expected total number of MAA, as well as the number, types and strength of MAA groups. These data depend on the effectiveness of aviation means' defeating ground targets and can also be estimated by using the appropriate models. However, the description of these models' research is beyond the scope of this consideration.

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