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Mathematical Models of Transonic Flutter of Aerodynamic Control Surfaces of Supersonic Aircraft

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Abstract:

In the article, the joint analysis of the Bernoulli equations for compressible gas, variations of the supersonic flow parameters of the Prandtl-Meyer expansion fan and the hypothesis of aerofoil dynamic curvature were used to develop linear and nonlinear mathematical models describing the occurrence of transonic flutter of aerodynamic control surfaces of supersonic aircraft. The analysis of the obtained mathematical models confirms a theoretical possibility of the occurrence of transonic flutter of aerodynamic control surfaces of supersonic aircraft which is due to the peculiarities of the interaction of shock waves with the angular velocity of elastic bending oscillations of aerodynamic control surfaces.

Keywords:

aerodynamic control surface, Mach number, mathematical model, oscillations, supersonic aircraft, transonic flutter

1 Introduction

Theoretical and experimental methods for studying classical (two-degree-of-freedom) flutter of aerodynamic surfaces in stationary and non-stationary flows have been developed quite thoroughly [1-4]. But so far, the problem of theoretical substantiation of the causes of intense oscillation of aerodynamic surfaces (“buzz”) of supersonic aircraft at transonic flight speeds remains unsolved. Thus, the authors [2, p. 621] indicate: “It appears impossible to predict this phenomenon quantitatively by classical aerodynamic theory, although the aileron’s motion is observed to be simply harmonic …”.

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A large number of scientific works [2-18] are devoted to the study of these oscillations, thanks to which three ranges of Mach numbers are determined where oscillations of aerodynamic surfaces are possible:

- range “A” – subsonic speed, the oscillations are caused due to the separation of the boundary layer behind the shock waves,
- range “B” – the aerofoil is in a mixed (subsonic and supersonic) airflow, the oscillations are due to the complex interaction of the shock waves with the oscillations of the aerodynamic control surfaces,
- range “C” – supersonic speed, the oscillations are possible on aerofoils with infinitesimal thickness.

The most dangerous are the oscillations of aerodynamic surfaces in the range “B”, the occurrence of which repeatedly led to severe flight accidents of supersonic aircraft. In this regard, some papers [6, 12, 13] refer to this phenomenon as the transonic flutter, a study of which is the purpose of the article.

2 Background

In the laboratory studies of oscillations of aerodynamic control surfaces it is noted that their occurrence is possible in the presence of only one degree of freedom [4, 5, 7], i.e. only at angular oscillations of control surfaces around their own axis.

The flight studies have shown that oscillations occur in a very narrow range of Mach numbers when the shock waves are near the trailing edge of the control surface. It has also been proven that the amplitude of oscillations increases with decreasing altitude, i.e. this type of oscillations can be attributed to nonlinear self-oscillations, whose amplitude depends on the characteristics of aerodynamic surfaces and flight conditions [6]. Moreover, cases of destruction of structural elements of aircraft were observed at relatively small amplitudes of oscillations of the control surfaces (1.5°-2.0°) [6, 18].

In theoretical works, based on the results of numerical calculations, the occurrence of this phenomenon is associated with the presence of air compression [8] and with the formation of shock waves on aerofoil surfaces [9]. According to [10], based on the results of numerical solution of the Navier–Stokes equations, it was concluded that the occurrence of oscillation is due to the phase delay of shock wave motion with regard to the oscillations of the control surface, but the causes of the phase delay were not disclosed.

In addition, as shown, for example, in [11], in the range of Mach numbers 0.95-1.1, numerical methods of calculation can lose their stability; therefore, the results of the studies obtained with the use of these methods should be treated with caution.

In [12], which is devoted to numerical methods for transonic flutter research, it is noted that, according to the results of experiments, in the near-sonic range of Mach numbers there is a significant decrease in critical velocity due to the motion of shock waves along the wing surface. Therefore, the study of this type of flutter using the methods of classical linear analysis of elastic oscillations is impossible.

In [13], which is also devoted to numerical methods of flutter research, it is indicated that flight safety of unmanned aerial vehicles in the range of Mach numbers 0.95-1.05 can be more accurately ensured by the results of flight tests.

The attitude to numerical methods is very categorically stated in [14], where it is indicated that all theoretical methods require simplification and reduction of the cost of calculations. Consequently, specialists in aeroelasticity must reveal the physical essence of the studied processes. Only then will the widespread use of computers and flight data be beneficial and will not turn into black magic [14].

It follows that there is still no generally accepted model for transonic flutter, so theoretical studies of transonic flutter of aerodynamic control surfaces of supersonic aircraft still remain a relevant scientific and applied problem.

The urgency of solving this problem is justified by the need to pre-assess the vibration level of aircraft structural elements before flight tests and the requirements to ensure flight safety of supersonic aircraft.

It also follows that the oscillations of aerodynamic control surfaces in the range "B" of Mach numbers can be attributed to the oscillations of elastic systems with one degree of freedom and can be represented by the mathematical model that was proposed in [15]:

$$\ddot{\delta}(t)+\frac{\vartheta}{\pi}\omega\dot{\delta}(t)+\omega^2\delta(t)=\frac{1}{\bar{J}_\mathrm{k}}\left[\bar{M}_\mathrm{a}\left(\delta;\dot{\delta}\right)+\bar{M}_\mathrm{c}\left(\delta;\dot{\delta}\right)\right] \tag{1}$$

where $\delta(t)$ – the deflection of the control surface under oscillation; ϑ – the logarithmic decrement of oscillation without taking into account aerodynamic damping; ω – the angular natural frequency of oscillation; $\bar{J}_\mathrm{k}$ – the distributed mass moment of inertia of the control surface; $\bar{M}_\mathrm{a}(\delta;\dot{\delta})$ – the distributed hinge moment of the control surface caused by aerodynamic forces, including aerodynamic damping forces; $\bar{M}_\mathrm{c}(\delta;\dot{\delta})$ – the distributed hinge moment of the control surface caused by shock waves.

Based on the equation (9.12) suggested in [2, p. 532], the distributed hinge moment of an aerodynamic control surface caused by aerodynamic forces can be represented by the equation

$$\bar{M}_\mathrm{a}\left(\delta,\dot{\delta}\right)=-\frac{1}{8}C_y^\delta\rho V^2 b_\mathrm{k}^2\delta(t)-\frac{3}{16}C_y^\delta\rho V b_\mathrm{k}^3\dot{\delta}(t) \tag{2}$$

where C_y^δ is the derivative of the lifting force coefficient with respect to the angle of deflection of the control surface; ρ – the air density at flight altitude; V – the flight speed; b_k – the aerofoil chord.

3 Analysis of Hinge Moments Caused by Shock Waves

When studying this phenomenon, the greatest difficulties arise in determining the hinge moments of the control surface caused by shock waves. In [16], these hinge moments were determined based on the mass flow rate equation, according to which the shock waves under oscillation of the control surface move to those intersections of the aerofoil chord where the thickness of the flow tube remains constant.

For the proposed mathematical models of transonic flutter of aerodynamic control surfaces, as in [16], the excited hinge moment of the control surface, i.e. the moment caused by shock waves, is determined by the dynamic curvature hypothesis [1, p. 95]. According to this hypothesis, the aerodynamic characteristics of an oscillating aerofoil are equal to the aerodynamic characteristics of a steady aerofoil curved in such a way that the local instantaneous angles of attack are defined by the equation

$$\Delta\alpha\left(x_\mathrm{c},\dot{\delta}\right)=\frac{x_\mathrm{c}\dot{\delta}(t)}{V} \tag{3}$$

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where x_c is the distance from the axis of rotation to the intersection of the aerofoil where the shock wave resides.

In addition, in mathematical models of transonic flutter of aerodynamic control surfaces, the excited hinge moment of the control surface caused by shock waves is determined taking into account the conditions of their formation [17]. These conditions are determined on the basis of the linearization of the Bernoulli equation for compressible fluid flow.

According to the Bernoulli equation for compressible fluid flow, the relative pressure of local supersonic flow on an aerofoil surface is defined by the ratio [19]:

$$\bar{P}_1 = \frac{P_1}{P_\infty} = \left(\frac{1 + \frac{\kappa - 1}{2} M_\infty^2}{1 + \frac{\kappa - 1}{2} M_1^2} \right)^{\frac{\kappa}{\kappa - 1}} \tag{4}$$

where P_1 is the pressure of local supersonic flow on the aerofoil surface; P_∞ – the pressure of undisturbed airflow; κ – the adiabatic index (for air $\kappa \approx 1{,}4$); M_∞ – the Mach number of undisturbed airflow; M_1 – the local Mach number of supersonic flow over the aerofoil surface.

Eq. (4) for thin aerofoils in a narrow range of Mach numbers of undisturbed airflow can be represented by an approximate linear equation, as was done in [17]:

$$\bar{P}_1 \approx 1 - M_1 + M_\infty \tag{5}$$

It should be noted that for $M_\infty = M_{\text{cr}}$ Eq. (5) turns into the following equation:

$$\bar{P}_1 = \bar{P}_{cr} \approx M_{cr} \tag{6}$$

where $\bar{P}_{\text{cr}}$ is the relative critical pressure of the local flow on the aerofoil surface, i.e. the relative pressure of the local flow on the aerofoil surface when $M_\infty = M_{\text{cr}}$ and $M_1 = 1.0$; M_{cr} – the critical Mach number of the aerofoil, i.e. the Mach number of undisturbed airflow at which shock waves occur on the aerofoil surface for the first time and the Mach number $M_1 = 1.0$.

When specifying $M_1 = 1.0$ and $M_\infty = M_{\text{cr}}$, the approximate Eq. (6) can be obtained directly from Eq. (4).

To determine how the relative pressure of local supersonic flow on an aerofoil surface depends on the Mach number of undisturbed subsonic airflow and the critical Mach number of the aerofoil, we use the fact that its adiabatic expansion begins at $M_\infty = M_{\text{cr}}$ when $M_1 = 1.0$. The nature of this dependence in the range of Mach numbers of undisturbed subsonic flow from the Mach number $M_\infty = M_{\text{cr}}$ to the Mach number $M_\infty = 1.0$ is also determined by the Bernoulli equation for compressible fluid flow, which in [17] is presented as:

$$\bar{P}_1 = \left(\frac{1 + \frac{\kappa - 1}{2} M_{\text{cr}}^2}{1 + \frac{\kappa - 1}{2} M_\infty^2} \right)^{\frac{\kappa}{\kappa - 1}} - (1 - \bar{P}_{\text{cr}}) \tag{7}$$

At complete adiabatic expansion, i.e. at the number $M_\infty = 1.0$, Eq. (7) is transformed into the following linear equation:

$$\Delta \bar{P}_1 = 1 - \bar{P}_1 = 2\left(1 - \bar{P}_{\text{cr}}\right) \tag{8}$$

Eq. (8) shows that with complete adiabatic expansion of local supersonic flow over an aerofoil surface, the maximum value of the pressure variation of the supersonic flow is twice its critical value.

Eq. (7), taking into account Eq. (5) and Eq. (6) together with Eq. (4), within the above-mentioned range of Mach numbers of undisturbed subsonic flow can also be represented as an approximate linear equation:

$$\bar{P}_1 \approx 2M_{\mathrm{cr}} - M_\infty \tag{9}$$

The acceptability of converting Eq. (7) into the approximate linear Eq. (9) can be confirmed in another way. Thus, in laboratory studies of this phenomenon [5, 7] it was proved that the magnitude of the pressure variation of local supersonic flow on an aerofoil surface increases monotonically with the increase of the Mach number of undisturbed airflow from $M_\infty = M_{\mathrm{cr}}$ to $M_\infty = 1.0$. This relationship, taking into account Eq. (6), can be described by an approximate linear equation:

$$\Delta\bar{P}_1 \approx \left(1 + \frac{M_\infty - M_{cr}}{1 - M_{cr}}\right)\left(1 - M_{\mathrm{cr}}\right) \tag{10}$$

Eq. (10) after the transformation can be expressed as Eq. (9). Eq. (9) and Eq. (10) at the Mach number $M_\infty = M_{\mathrm{cr}}$ turn into Eq. (6), and at the Mach number $M_\infty = 1.0$, taking into account Eq. (6), into Eq. (8).

From the comparison of Eq. (5) and Eq. (9) the following expression can be obtained:

$$M_1 - 1 \approx 2\left(M_\infty - M_{\mathrm{cr}}\right) \tag{11}$$

Eq. (11) is a condition for the transition of local supersonic flow to subsonic flow or a condition for the formation of shock waves on an aerofoil surface.

Eq. (11) is also a condition for adiabatic expansion of local supersonic flow over the aerofoil surface. It shows that when the Mach number of undisturbed airflow increases from $M_\infty = M_{\mathrm{cr}}$ to $M_\infty = 1.0$, the increase of the Mach number of local supersonic flow over an aerofoil surface is twice as large as the increase of the Mach number of undisturbed airflow.

An analogue of the obtained Eq. (8) and Eq. (11) can be observed in the mechanics of elastic systems. Namely, the deflection of an undamped elastic system under the step excitation is twice the deflection under the action of a static force.

It also follows from Eq. (11) that the Mach number of local supersonic flow over the aerofoil surface cannot exceed the value

$$M_1 \leq 1 + 2\left(M_\infty - M_{\mathrm{cr}}\right) \tag{12}$$

At the same time, as follows from the analysis of the properties of the Prandtl-Meyer expansion fan, the Mach number of local supersonic flow is determined only by the angle of its deflection [19]. This approximate relation, as in [17], can be represented as

$$M_1 \approx \sqrt[3]{1 + 11.5\varphi(x)} \tag{13}$$

where $\varphi(x)$ is the angle of deflection of local supersonic flow over the aerofoil chord of an aerodynamic surface.

It should be noted that the error of the approximate Eqs (5), (6), (9), (11), and (13) for aerofoils with a relative thickness 0.04-0.08 does not exceed 1.0 %-2.0 %.

The analysis of Eqs (12) and (13) allows to substantiate patterns of interaction of shock waves with oscillations of control surfaces.

Thus, the Eq. (13) shows that at $\varphi(x) = 0$ the Mach number of local supersonic flow over the surface of an aerofoil even under the condition $M_\infty > M_{cr}$ cannot exceed the value $M_1 = 1.0$. When $\varphi(x) > 0$ the Mach number M_1 increases and the intensity of the shock waves also increases, but the value of the Mach number M_1 cannot exceed the value determined by the expression (12), even at large angles of deflection of the local supersonic flow. This can explain the fact of increasing intensity of shock waves with their motion forward to the axis of rotation of the control surface, which was observed in laboratory studies [7].

When the Mach number M_∞ = const and the Mach number M_{cr} = const, then, as follows from the condition (12), the Mach number M_1 of local supersonic flow over the surface of an aerofoil remains constant. Moreover, the Mach number M_1 also remains practically constant when the amplitude of oscillations of the control surface is low, i.e. under the condition

$$\varphi(x) = \varphi(x_i) \pm \delta_0 \tag{14}$$

where $\varphi(x_i)$ is the inclination angle of the tangent to the aerofoil in the cross section of the chord x_i; δ_0 is the amplitude of oscillations of the control surface.

It can be concluded from the above that when a control surface is deflected by an angle the value of which is limited by Eq. (14) the shock waves move from the initial location back and forth to those cross sections of the control surface aerofoil chord where the deflection angle of the local supersonic flow is equal to the deflection angle in the cross section of the initial location. That is, the shock waves move to those cross sections of the chord where, according to Eq. (13), the Mach number M_1 remains constant.

These patterns of the interaction of shock waves with the deflection of the aerodynamic control surface can be described as follows:

- when shock waves move forward to the axis of rotation from the initial location on the control surface

$$\varphi(x) = \varphi(x - \Delta x_{\mathrm{f}}) + \delta_0 \tag{15}$$

- when shock waves move back from the initial location on the control surface

$$\varphi(x) = \varphi(x + \Delta x_{\mathrm{b}}) - \delta_0 \tag{16}$$

In Eqs (15) and (16) the following notations are introduced: Δx_{f} – the distance of shock waves motion forward from the initial location on the control surface; Δx_{b} – the distance of shock waves motion back from the initial location on the control surface.

From the analysis of Eqs (13), (15) and (16) we see that to assess the nature of the interaction of shock waves with the oscillations of control surfaces it is necessary to know the geometrical characteristics of aerofoils, namely, the nature of variation of the inclination angle of the tangent to an aerofoil behind the chord.

For an approximate quantitative evaluation of the geometric characteristics of aerofoils of modern supersonic aircraft, this relation can be represented as a linear equation

$$\varphi(x) \approx \frac{\varphi_0}{b_1} x_i \tag{17}$$

where φ_0 is the maximum inclination angle of the tangent to an aerofoil (near the trailing edge); b_1 – the distance from the line of the maximum thickness of the aerofoil to its trailing edge; x_i – the distance from the intersection i of the aerofoil chord to the line of the maximum thickness of the aerofoil.

Taking into account Eqs (17), (15) and (16) leads to the following relation:

$$\Delta x_{\mathrm{f}} = \Delta x_{\mathrm{b}} \approx b_1 \frac{\delta_0}{\varphi_0} \tag{18}$$

According to the results of the experimental studies [6], it is known that the speed of shock waves motion during oscillation is 1.0 %-4.0 % of the speed of sound. That is, when a control surface oscillates, Eq. (18) can be represented as

$$\Delta x_{\mathrm{f}}\left(\delta;t\right) = \Delta x_{\mathrm{b}}\left(\delta;t\right) \approx \frac{b_1}{\varphi_0}\delta\left(t\right) \tag{19}$$

The displacements of shock waves that are defined by Eqs (18) and (19) cause such location of the local flow on the control surface when a destabilizing hinge moment occurs, i.e. the moment that is directed towards the deflection of the control surface.

The distributed value of this moment can be determined by the integral

$$\bar{M}_1\left(x_{\mathrm{c}};\delta\right) = \int_{x_1}^{x_2}\left(P_{\mathrm{L}} - P_{\mathrm{U}}\right)x_{\mathrm{c}}\mathrm{d}x \tag{20}$$

where x_{c} is the distance from the initial location of shock waves to the axis of rotation of the control surface in the absence of oscillations; P_{U} – the average pressure of the local flow on the part of the upper surface of the aerofoil where shock waves move along; P_{L} – the average pressure of the local flow on the part of the lower surface of the aerofoil where shock waves move along; x_1 – the minimum distance from the shock waves to the axis of rotation of the control surface during oscillation; x_2 – the maximum distance from the shock waves to the axis of rotation of the control surface during oscillation.

At small amplitudes of oscillations of control surfaces, which can include oscillations under transonic flutter, the integral (20) can be represented by the following approximate equation

$$\bar{M}_1\left(x_{\mathrm{c}};\delta\right) \approx \frac{1}{2}\Delta P_{\mathrm{c}}\left(x_2^2 - x_1^2\right) \approx \frac{1}{2}\Delta P_{\mathrm{c}}\left(x_2 - x_1\right)\left(x_2 + x_1\right) \tag{21}$$

where ΔP_{c} is the average variation of local flow pressure on the lower and upper control surfaces on the part of an aerofoil where shock waves move along.

Eq. (21) can also be represented as

$$\bar{M}_1\left(x_{\mathrm{c}};\delta\right) \approx \frac{1}{2}\Delta P_{\mathrm{c}}\left\{\left[\left(x_{\mathrm{c}} + \Delta x_{\mathrm{b}}\right) - \left(x_{\mathrm{c}} - \Delta x_{\mathrm{f}}\right)\right]\left[\left(x_{\mathrm{c}} + \Delta x_{\mathrm{b}}\right) + \left(x_{\mathrm{c}} - \Delta x_{\mathrm{f}}\right)\right]\right\} \tag{22}$$

Substituting Eq. (19) into Eq. (22) we obtain

$$\bar{M}_1\left(x_{\mathrm{c}};\delta\right) \approx 2\Delta P_{\mathrm{c}} x_{\mathrm{c}} \frac{b_1}{\varphi_0}\delta\left(t\right) \tag{23}$$

From Eq. (23) it follows that the destabilizing hinge moment increases when the amplitude of oscillations of the control surface increases and when the distance of displacement of the shock waves to the trailing edge of the aerofoil increases. This relation is observed only when the motion of shock waves is not limited by the trailing edge of the aerofoil. This limitation is determined by the following condition:

$$x_{\mathrm{c}} \geq b_{\mathrm{k}} - \Delta x_{\mathrm{b}}(\delta;t) \tag{24}$$

On this part of the aerofoil chord, i.e. under the condition (24), the total displacement of the shock waves forward and backward from the initial location decreases as follows:

$$l(x_{\mathrm{c}};\delta) = b_{\mathrm{k}} - x_{\mathrm{c}} + \Delta x_{\mathrm{f}}(\delta;t) \tag{25}$$

Given Eq. (19), Eq. (25) can be represented as

$$l(x_{\mathrm{c}};\delta) = b_{\mathrm{k}} - x_{\mathrm{c}} + \frac{b_1}{\varphi_0}\left|\delta(t)\right| \tag{26}$$

The distributed hinge moment of the control surface on this part of the location of shock waves can be represented by a relation similar to Eq. (21) with the following substitutions:

$$x_1 = b_{\mathrm{k}} - l(x_{\mathrm{c}};\delta) \tag{27}$$

$$x_2 = b_{\mathrm{k}} \tag{28}$$

Substituting Eqs (27) and (28) into Eq. (21) and taking into account Eq. (26), we obtain the relation that describes the decrease in the destabilizing hinge moment at this part of the location of shock waves

$$\bar{M}_2(x_{\mathrm{c}};\delta) \approx \frac{1}{2}\Delta P_{\mathrm{c}}\left[b_{\mathrm{k}} - x_{\mathrm{c}} + \frac{b_1}{\varphi_0}\left|\delta(t)\right|\right]\left[b_{\mathrm{k}} + x_{\mathrm{c}} - \frac{b_1}{\varphi_0}\left|\delta(t)\right|\right] \tag{29}$$

The maximum value of the distributed destabilizing hinge moment of the control surface will be under the condition (24). Substituting it into Eq. (23) or Eq. (29) we obtain

$$\bar{M}_{\mathrm{c}}(\delta) \approx 2\Delta P_{\mathrm{c}}\left[b_{\mathrm{k}} - \frac{b_1}{\varphi_0}\left|\delta(t)\right|\right]\frac{b_1}{\varphi_0}\delta(t) \tag{30}$$

The occurrence of destabilizing hinge moments of control surfaces at transonic flight speeds was repeatedly observed in laboratory studies [20, pp. 403, 404]. This phenomenon is called "reversal control", but it causes only deflection, not oscillation of control surfaces.

The oscillation of control surfaces is caused by the excited hinge moment. To determine this moment, as mentioned above, it is necessary to use the hypothesis of dynamic curvature.

In this case, to determine the pattern of interaction of shock waves with oscillations of control surfaces, Eqs (15) and (16), which define the conditions for interaction of shock waves with the deflection of control surfaces, should be represented taking into account the local instantaneous angles of attack, which are defined by Eq. (3):

- when the shock waves move forward (to the axis of rotation) from the initial location

$$\varphi(x)=\varphi\left[x-\Delta x_{\mathrm{f}}\left(\dot{\delta}\right)\right]+\frac{x_{\mathrm{c}}-\Delta x_{\mathrm{f}}\left(\dot{\delta}\right)}{V}\dot{\delta}(t) \tag{31}$$

- when the shock waves move backward from the initial location

$$\varphi(x)=\varphi\left[x+\Delta x_{\mathrm{b}}\left(\dot{\delta}\right)\right]-\frac{x_{\mathrm{c}}+\Delta x_{\mathrm{b}}\left(\dot{\delta}\right)}{V}\dot{\delta}(t) \tag{32}$$

In Eq. (31) and Eq. (32) the following notations are introduced: $\Delta x_{\mathrm{f}}(\dot{\delta})$ – the displacement of shock waves caused by the deflection speed of a control surface, forward from the initial location; $\Delta x_{\mathrm{b}}(\dot{\delta})$ – the displacement of shock waves caused by the deflection speed of a control surface, back from the original location.

After the transformation of Eqs (31), (32), taking into account Eq. (17), we obtain:

$$\Delta x_{\mathrm{f}}\left(\dot{\delta}\right)=\frac{x_{\mathrm{c}}b_1\dot{\delta}(t)}{\varphi_0 V+b_1\left|\dot{\delta}(t)\right|} \tag{33}$$

$$\Delta x_{\mathrm{b}}\left(\dot{\delta}\right)=\frac{x_{\mathrm{c}}b_1\dot{\delta}(t)}{\varphi_0 V-b_1\left|\dot{\delta}(t)\right|} \tag{34}$$

The interaction of the shock waves with the oscillations of the control surfaces, taking into account both their deflections and the speed of their deflections, is determined by the sum of Eqs (19), (33), and (34)

$$\Delta x\left(\delta;\dot{\delta}\right)=\frac{b_1}{\varphi_0}\delta(t)+\frac{x_{\mathrm{c}}b_1\dot{\delta}(t)}{\varphi_0 V\pm b_1\left|\dot{\delta}(t)\right|} \tag{35}$$

Visualization of Eq. (35) is presented in Fig. 1.

The displacements of shock waves, which are defined by Eqs (33) and (34), change the location of the local flow pressure on the control surface, and as a consequence, they cause an excited hinge moment, i.e. the moment acting towards the deflection of the control surface.

The distributed excited hinge moment of the control surface is defined by an equation that is similar to Eq. (22):

$$\bar{M}_1\left(x_{\mathrm{c}};\dot{\delta}\right)\approx\frac{1}{2}\Delta P_{\mathrm{c}}\left\{\left[x_{\mathrm{c}}+\Delta x_{\mathrm{b}}\left(\dot{\delta}\right)\right]-\left[x_{\mathrm{c}}-\Delta x_{\mathrm{f}}\left(\dot{\delta}\right)\right]\right\}\left\{\left[x_{\mathrm{c}}+\Delta x_{\mathrm{b}}\left(\dot{\delta}\right)\right]+\left[x_{\mathrm{c}}-\Delta x_{\mathrm{f}}\left(\dot{\delta}\right)\right]\right\} \tag{36}$$

Substituting Eqs (33) and (34) into (36), we obtain an approximate relation for variation of the distributed excited hinge moment dependent on the amplitude of oscillations of the control surface and the location of the shock waves on the aerofoil chord:

$$\bar{M}_1\left(x_{\mathrm{c}};\dot{\delta}\right)\approx\frac{2\Delta P_{\mathrm{c}}x_{\mathrm{c}}^2b_1\dot{\delta}(t)}{\left[\varphi_0^2V^2-b_1^2\dot{\delta}^2(t)\right]^2} \tag{37}$$

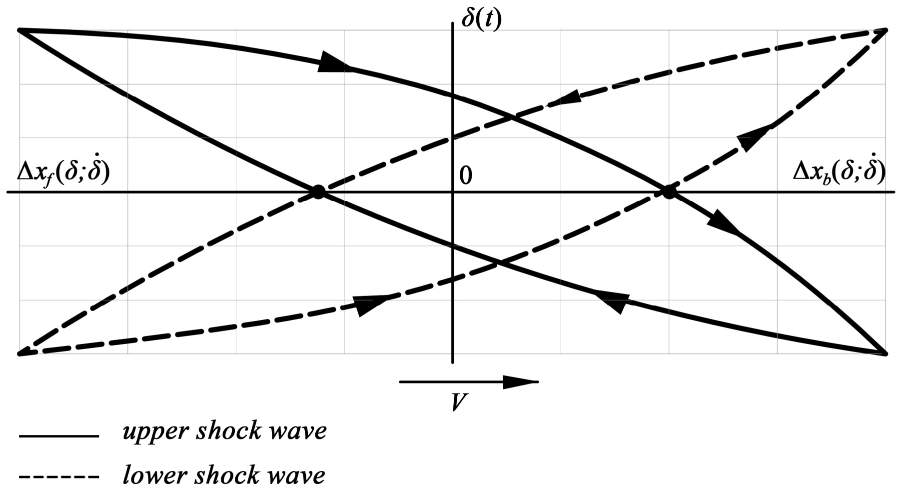

Fig. 1 The interaction of oscillations of shock waves with oscillations of the aerodynamic control surface of the keel

From the analysis of Eq. (37) it follows that when the level of oscillations increases or when the shock waves approach the trailing edge of the control surface the excited hinge moment increases. But this pattern is observed only when the motion of shock waves under oscillation of the control surface is not limited by the trailing edge of the aerofoil. As above, this limitation is determined by a condition similar to the condition (24):

$$x_{\mathrm{c}}\left(\dot{\delta}\right) \geq b_{\mathrm{k}} - \Delta x_{\mathrm{b}}\left(\dot{\delta}\right) \tag{38}$$

On this part of the aerofoil chord, when the control surface oscillates, the total displacement of the shock waves back and forth from the initial location decreases, which can be described by the following equation:

$$l\left(x_{\mathrm{c}};\dot{\delta}\right) = b_{\mathrm{k}} - x_{\mathrm{c}} + \Delta x_{\mathrm{f}}\left(\dot{\delta}\right) \tag{39}$$

Substituting Eq. (33) into Eq. (39), we obtain a relation for variation of the total displacement of shock waves on this part of the control surface chord:

$$l\left(x_{\mathrm{c}};\dot{\delta}\right) = b_{\mathrm{k}} - \frac{x_{\mathrm{c}}\varphi_0 V}{\varphi_0 V + b_1\left|\dot{\delta}(t)\right|} \tag{40}$$

The distributed excited hinge moment of the control surface on this part of the location of the shock waves can be represented by a relation similar to Eq. (21) where the variables x_1 and x_2 are substituted by the following equations:

$$x_1 = b_{\mathrm{k}} - l\left(x_{\mathrm{c}};\dot{\delta}\right) \tag{41}$$

$$x_2 = b_{\mathrm{k}} \tag{42}$$

Substituting Eq. (41) and Eq. (42) into Eq. (21) and taking into account Eq. (40), we obtain a relation for the decrease of the distributed excited hinge moment on this part of the location of the shock waves:

$$\bar{M}_2\left(x_{\mathrm{c}};\dot{\delta}\right) \approx \frac{1}{2}\Delta P_{\mathrm{c}}\left\{b_{\mathrm{k}}^2 - \frac{x_{\mathrm{c}}^2\varphi_0^2V^2}{\left[\varphi_0 V + b_1\left|\dot{\delta}(t)\right|\right]^2}\right\} \tag{43}$$

The maximum value of the distributed excited hinge moment of the control surface will be under the condition (38). Substituting the condition (38) into Eq. (37) or Eq. (43) and taking into account Eq. (34), we obtain

$$\bar{M}_{\mathrm{c}}\left(\dot{\delta}\right) \approx 2\Delta P_{\mathrm{c}} b_{\mathrm{k}}^2 \frac{b_1\varphi_0 V\dot{\delta}(t)}{\left[\varphi_0 V + b_1\left|\dot{\delta}(t)\right|\right]^2} \tag{44}$$

It is the moment which causes intense oscillations of the aerodynamic control surfaces of supersonic aircraft at transonic flight speeds.

From the results obtained above, it follows that for low-amplitude harmonic oscillations of control surfaces the distributed hinge moment caused by the shock waves can be represented as an approximate linear function, which is the vector sum of Eq. (30) and Eq. (44) simplified by linear functions [21]:

$$\bar{M}_{\mathrm{c}}\left(\delta;\dot{\delta}\right) \approx 2\Delta P_{\mathrm{c}} b_{\mathrm{k}} b_1\sqrt{1+\bar{\omega}^2}\,\frac{\delta_0}{\varphi_0}\sin\left(\omega t + \alpha\right) \tag{45}$$

In Eq. (45) the following notations are introduced:

- dimensionless angular frequency of oscillations of the control surface (the Strouhal number),

$$\bar{\omega} = \frac{\omega b_{\mathrm{k}}}{V}$$

- phase advance angle of the hinge moment in relation to the angle of deflection of the control surface

$$\alpha = \operatorname{arctg}\bar{\omega}$$

4 Linear Model of Transonic Flutter

From the analysis of Eq. (30), Eq. (44) and Eq. (45) it follows that the phase advance angle is due only to the presence of the excited hinge moment of the control surface.

In flight studies of this phenomenon [6] it is noted that when the flight altitude varies, the oscillation frequencies, as a rule, remain practically constant and are equal to the natural frequencies of low tones of elastic oscillations of the control surfaces on the ground. That is, the values of the destabilizing hinge moments are equal to the values of the aerodynamic hinge moments of the control surfaces. Therefore, a linear mathematical model of the occurrence of transonic flutter of aerodynamic control surfaces of supersonic aircraft can be presented without taking into account these hinge moments, in the following form:

$$\ddot{\delta}(t) + \frac{\vartheta}{\pi}\omega\dot{\delta}(t) + \omega^2\delta(t) = \frac{1}{\bar{J}_{\mathrm{k}}}\left\{-\frac{3}{16}C_y^{\delta}\rho V b_{\mathrm{k}}^3\dot{\delta}(t) + 2\Delta P_{\mathrm{c}} b_{\mathrm{k}}^2\frac{b_1}{\varphi_o V}\dot{\delta}(t)\right\} \tag{46}$$

The linear mathematical model of transonic flutter of aerodynamic control surfaces (46) reflects the causes for the formation of excited hinge moments of the control surfaces and confirms the possibility of transonic flutter occurrence under the only one degree of freedom, namely, when control surfaces oscillate around their axis of rotation.

5 Nonlinear Model of Transonic Flutter

However, to assess the level of oscillation of the control surfaces it is necessary to consider the following. From the analysis of the results of experimental studies and Eq. (30) and Eq. (44) it follows that these oscillations belong to the type of nonlinear oscillations with limited amplitude.

In addition, as follows from the above, the excited hinge moment of the control surface reaches its maximum value under the condition (38), when the value of the destabilizing hinge moment is lower than the value determined by Eq. (30).

That is, the mathematical model of transonic flutter of aerodynamic control surfaces should take into account the nonlinear nature of the excited hinge moment, which is determined by Eq. (44), and the nonlinear nature of the destabilizing hinge moment, which should be determined by Eq. (29) taking into account the condition (38) but not the condition (24). This significantly complicates the development of a nonlinear mathematical model of transonic flutter of aerodynamic control surfaces of supersonic aircraft.

But, as noted above, since under the occurrence of transonic flutter the oscillation frequencies of the control surfaces remain stable, then a nonlinear mathematical model of transonic flutter of aerodynamic control surfaces can be formed without taking into account the destabilizing hinge moments of the control surfaces, which are caused by the shock waves, and the hinge moments from the aerodynamic forces.

On the other hand, in flight studies of supersonic aircraft a very high sensitivity of the level of oscillations of the control surfaces to various characteristics was repeatedly observed, first of all, to the geometric characteristics of aerofoils. Namely, the level of oscillations of control surfaces under the same flight modes of the same type of aircraft is not always the same. Therefore, when substantiating a mathematical model of transonic flutter of aerodynamic control surfaces of supersonic aircraft, it is necessary to take into account the nature of pressure variation behind the aerofoil chord, as follows from the analysis of Eq. (10) and Eq. (13).

Given Eq. (8) and the almost linear nature of pressure variation behind the aerofoil chord, which is confirmed by the results of laboratory studies [22], this relation in [17] is presented in the following form:

$$\Delta P(x) \approx \frac{1}{2} \Delta P_0 \left(1 + \frac{b_0 + x_c}{b_1} \right) \tag{47}$$

where ΔP_0 – the maximum value of variation of local supersonic flow pressure on the surface of an aerofoil when shock waves are located near the trailing edge of the aerofoil and there are no oscillations; b_0 – the distance from the axis of rotation of the control surface to the line of maximum thickness of the aerofoil of the bearing surface.

Eq. (47) after the transformation can be represented as follows [17]:

$$\Delta P(x) \approx \Delta P_0 \left[1 - \frac{1}{2} \frac{b_k}{b_1} \left(1 - \frac{x_c}{b_k} \right) \right] \tag{48}$$

In addition, from the analysis of the results of laboratory studies that are presented in [22], it follows that the pressure behind the shock waves varies from the value P_1

to the value P_∞ approximately linearly. Therefore, the distributed value of the excited force caused by the shock waves can be represented as

$$\bar{F}_1\left(x_c;\dot{\delta}\right) \approx \frac{1}{2}\Delta P(x) l_1\left(x_c;\dot{\delta}\right) \tag{49}$$

where $l_1(x_c;\dot{\delta})$ – the total value of displacement of the shock waves back and forth from the original location.

This displacement can be represented by the sum of Eq. (33) and Eq. (34):

$$l_1\left(x_c;\dot{\delta}\right) = \Delta x_f\left(\dot{\delta}\right) + \Delta x_b\left(\dot{\delta}\right) = \frac{2x_c b_1 \varphi_0 V \dot{\delta}(t)}{\varphi_0^2 V^2 - b_1^2 \dot{\delta}^2(t)} \tag{50}$$

Taking into account Eqs (48)-(50), the distributed value of the force caused by the shock waves can be defined by the following equation:

$$\bar{F}_1\left(x_c;\dot{\delta}\right) \approx \Delta P_0\left[1 - \frac{1}{2}\frac{b_k}{b_1}\left(1 - \frac{x_c}{b_k}\right)\right]\frac{x_c b_1 \varphi_0 V \dot{\delta}(t)}{\varphi_0^2 V^2 - b_1^2 \dot{\delta}^2(t)} \tag{51}$$

At small amplitudes of oscillations of control surfaces, which can include oscillations of control surfaces in case of transonic flutter, the relation between the distributed excited hinge moment of the control surface and the location of the shock waves and the amplitude of oscillations can be given as follows

$$\bar{M}_1\left(x_c;\dot{\delta}\right) \approx \bar{F}_1\left(x_c;\dot{\delta}\right)\left[x_c + \frac{\Delta x_b\left(\dot{\delta}\right) - \Delta x_f\left(\dot{\delta}\right)}{2}\right] \tag{52}$$

Substituting Eq. (33) and Eq. (34) into Eq. (52), we obtain

$$\bar{M}_1\left(x_c;\dot{\delta}\right) \approx \Delta P_0\left[1 - \frac{1}{2}\frac{b_k}{b_1}\left(1 - \frac{x_c}{b_k}\right)\right]\frac{x_c^2 b_1 \varphi_0^3 V^3 \dot{\delta}(t)}{\left[\varphi_0^2 V^2 - b_1^2 \dot{\delta}^2(t)\right]^2} \tag{53}$$

The maximum value of the distributed excited hinge moment of the control surface is obtained by substituting the condition (38) into Eq. (53), taking into account the pressure variation of the local supersonic flow on the surface of the aerofoil:

$$\bar{M}_1\left(\dot{\delta}\right) \approx \Delta P_0\left[1 - \frac{1}{2}\frac{b_k}{\varphi_0 V}\left|\dot{\delta}(t)\right|\right]\frac{b_k^2 b_1 \varphi_0 V \dot{\delta}(t)}{\left[\varphi_0 V + b_1\left|\dot{\delta}(t)\right|\right]^2} \tag{54}$$

As follows from the analysis of Eq. (54), the relation between the maximum value of the distributed excited hinge moment and the amplitude of oscillations of the control surface is nonlinear, so the mathematical model of transonic flutter of aerodynamic control surfaces of supersonic aircraft is also nonlinear. The model, taking into account the above mentioned, can be represented by the following differential equation with a nonlinear right-hand side:

$$\begin{gathered}\ddot{\delta}(t) + \frac{\vartheta}{\pi}\omega\dot{\delta}(t) + \omega^2\delta(t) = \\ = \frac{1}{\bar{J}_k}\left\{-\frac{3}{16}C_y^\delta \rho V b_k^3 \dot{\delta}(t) + \Delta P_0\left[1 - \frac{1}{2}\frac{b_k}{\varphi_0 V}\left|\dot{\delta}(t)\right|\right]\frac{b_k^2 b_1 \varphi_0 V}{\left[\varphi_0 V + b_1\left|\dot{\delta}(t)\right|\right]^2}\dot{\delta}(t)\right\}\end{gathered} \tag{55}$$

6 Conclusions

The obtained mathematical models of transonic flutter of aerodynamic control surfaces of supersonic aircraft highlight the main physical processes caused by the interaction of shock waves with the oscillations of aerodynamic control surfaces and define the peculiarities of this phenomenon observed in laboratory and flight studies. They confirm the theoretical possibility of the occurrence of transonic flutter of aerodynamic control surfaces the oscillations of which can be represented by the oscillations of elastic systems with one degree of freedom.

These mathematical models can be used for preliminarily assessment of the conditions of occurrence of transonic flutter of aerodynamic control surfaces and the flutter's characteristics when determining the safe conditions of flight operation of supersonic aircraft as well as analyzing the causes of flight accidents of supersonic aircraft.

The possibility of approximate estimation of some characteristics of transonic flutter of aerodynamic control surfaces using the nonlinear mathematical model was confirmed by comparing them with the corresponding characteristics obtained in flight experiments [23, 24].

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